Appendix to "War Chests as Precautionary Savings"

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A Proofs of Propositions

In this appendix, I present the solution to the incumbent's maximization problem. The method is to work backwards through each decision point of the incumbent solving each maximization problem with the Kuhn-Tucker method. One of the insights one obtains is that the incumbent would raise a certain amount if she knew who she was running against. But since she does not know, she raises some "average" amount. This amount is different than what the incumbent would raise if she knew who she was running against. Finally, I show the comparative statics of war chests and other variables by taking partial derivatives.

Because $W_1(s,q) > 0$,[1] the incumbent will spend all that she has in the last election: $s_2 = r_1 - s_1 + r_2$ (recall that subscripts denote the first or second election). Since the probabilities of drawing a particular quality challenger are known by the incumbent, the incumbent's utility maximization problem is a nested maximization problem:

$$\max_{r_1 \geq 0} \left\langle \begin{array}{l} -C(r_1)+ \\ \max\limits_{0 \leq s_1^{q_1} \leq r_1} \int_{Q_1} f(q_1) W(s_1^{q_1}, q_1) dq_1 \left\{ 1 + \max\limits_{r_2^{q_1} \geq 0} \left[-C(r_2^{q_1}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1} + r_2^{q_1}, q_2) dq_2 \right] \right\}. \end{array} \right\rangle$$

(Recall that the superscript q_1 refers to the quality of challenger the incumbent faced in the first election.) I solve the inner maximization problems and work outwards (to the overall maximization problem).

A.1 Solving for money raised in the second election

First solve:

$$\max_{r_2^{q_1} \geq 0} \left[-C(r_2^{q_1}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1} + r_2^{q_1}, q_2) dq_2 \right].$$

For this problem, $r_1 - s_1^{q_1}$ (the war chest) is a constant. Using the method of Kuhn-Tucker to solve this problem, I first set up the Lagrangian:

$$L(r_2^{q_1}, \lambda) = -C(r_2^{q_1}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1} + r_2^{q_1}, q_2) dq_2 + \lambda r_2^{q_1}.$$

Using Leibniz's rule, the first-order conditions are

$$\frac{\partial L}{\partial r_2^{q_1}} = -C_1(r_2^{q_1}) + \int_{Q_2} f(q_2) W_1(r_1 - s_1^{q_1} + r_2^{q_1}, q_2) dq_2 + \lambda = 0$$

$$\lambda \geq 0, \quad r_2^{q_1} \geq 0, \quad \lambda r_2^{q_1} = 0.$$

There are two cases: 1-$\{\lambda > 0,\ r_2^{q_1} = 0\}$and 2-$\{\lambda = 0,\ r_2^{q_1} > 0\}$. I examine the cases separately.

[1]Following convention, a subscript n on a function denotes the partial derivative of that function with respect to the nth argument of the function. Thus, $W_1(s,q) = \dfrac{\partial W(s,q)}{\partial s}$.

A.1.1 Case 1:

Here $r_2^{q_1} = 0$. Therefore,

$$\int_{Q_2} f(q_2) W_1(r_1 - s_1^{q_1} + r_2^{q_1}, q_2) dq_2 < C_1(0).$$

Since $C_1(0) = 0$ and $W_1(s, q) > 0$ by assumption, this cannot be true.

A.1.2 Case 2:

Here $\lambda = 0$. For a given war chest $(r_1 - s_1^{q_1})$, find the $r_2^{q_1} > 0$ that solves

$$\int_{Q_2} f(q_2) W_1(r_1 - s_1^{q_1} + r_2^{q_1}, q_2) dq_2 = C_1(r_2^{q_1}). \tag{1}$$

For future use I derive some comparative statics. Let $r_2^{q_1 *} = r_2^{q_1 *}(r_1 - s_1^{q_1}, q_1)$ be the $r_2^{q_1}$ that solves Equation (1). Substituting $r_2^{q_1 *}$, I obtain

$$\int_{Q_2} f(q_2) W_1(r_1 - s_1^{q_1} + r_2^{q_1 *}, q_2) dq_2 \equiv C_1(r_2^{q_1 *}). \tag{2}$$

Partially differentiating both sides with respect to $s_1^{q_1}$, I obtain

$$\left\{ \int_{Q_2} f(q_2) W_{11}(r_1 - s_1^{q_1} + r_2^{q_1 *}, q_2) dq_2 \left[-1 + \frac{\partial r_2^{q_1 *}}{\partial s_1^{q_1}} \right] \right\} = C_{11}(r_2^{q_1 *}) \frac{\partial r_2^{q_1 *}}{\partial s_1^{q_1}}.$$

Since $W_{11}(s, q) < 0$ and $C_{11}(r) > 0$, I find that $0 < \dfrac{\partial r_2^{q_1 *}}{\partial s_1^{q_1}} < 1$. Looking further ahead, I substitute the $r_2^{q_1 *}$ that solves Equation (1) with the equilibrium $s_1^{q_1}$ $(s_1^{q_1 *} = s_1^{q_1 *}(r_1, q_1))$:

$$\int_{Q_2} f(q_2) W_1(r_1 - s_1^{q_1 *} + r_2^{q_1 *}, q_2) dq_2 \equiv C_1(r_2^{q_1 *}). \tag{3}$$

Differentiating both sides with respect to r_1, I obtain

$$\left\{ \int_{Q_2} f(q_2) W_{11}(r_1 - s_1^{q_1 *} + r_2^{q_1 *}, q_2) dq_2 \left[1 - \frac{d s_1^{q_1 *}}{d r_1} + \frac{d r_2^{q_1 *}}{d r_1} \right] \right\} = C_{11}(r_2^{q_1 *}) \frac{d r_2^{q_1 *}}{d r_1}.$$

Since $W_{11}(s, q) < 0$ and $C_{11}(r) > 0$, I find

$$\begin{aligned} \frac{d r_2^{q_1 *}}{d r_1} > 0 \quad &\Leftrightarrow \quad \frac{d s_1^{q_1 *}}{d r_1} > 1 + \frac{d r_2^{q_1 *}}{d r_1} > 1 \\ \frac{d r_2^{q_1 *}}{d r_1} = 0 \quad &\Leftrightarrow \quad \frac{d s_1^{q_1 *}}{d r_1} = 1 \\ \frac{d r_2^{q_1 *}}{d r_1} < 0 \quad &\Leftrightarrow \quad \frac{d s_1^{q_1 *}}{d r_1} < 1 + \frac{d r_2^{q_1 *}}{d r_1} < 1. \end{aligned} \tag{4}$$

A.2 Solving for the money spent in the first election

Now, solve the next maximization problem—finding $s_1^{q_1}$:

$$\max_{0 \le s_1^{q_1} \le r_1} W(s_1^{q_1}, q_1) \left[1 - C(r_2^{q_1 *}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1} + r_2^{q_1 *}, q_2) dq_2 \right].$$

In this problem, r_1 is treated as a constant. Solve again through the method of Kuhn-Tucker by first setting up the Lagrangian:

$$\begin{aligned} L(s_1^{q_1}, \mu_1, \mu_2) = \quad & W(s_1^{q_1}, q_1)\left\{1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1} + r_2^{q_1*}, q_2) dq_2\right\} \\ & + \mu_1 s_1^{q_1} + \mu_2 (r_1 - s_1^{q_1}). \end{aligned}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial s_1^{q_1}} = \quad & W_1(s_1^{q_1}, q_1)\left\{1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1} + r_2^{q_1*}, q_2) dq_2\right\} \\ & + W(s_1^{q_1}, q_1)\left\{-C_1(r_2^{q_1*})\frac{\partial r_2^{q_1*}}{\partial s_1^{q_1}} + \left[\int_{Q_2} f(q_2) W_1(r_1 - s_1^{q_1} + r_2^{q_1*}, q_2) dq_2\right]\left[-1 + \frac{\partial r_2^{q_1*}}{\partial s_1^{q_1}}\right]\right\} \\ & + \mu_1 - \mu_2 \\ = \quad & 0 \end{aligned}$$

$$\mu_1 s_1^{q_1} = 0, \quad \mu_1 \geq 0, \quad s_1^{q_1} \geq 0$$
$$\mu_2 (r_1 - s_1^{q_1}) = 0, \quad \mu_2 \geq 0, \quad r_1 \geq s_1^{q_1}.$$

Using Equation (2), substitute and simplify, and obtain

$$\begin{aligned} W_1(s_1^{q_1}, q_1)\left\{1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1} + r_2^{q_1*}, q_2) dq_2\right\} & \\ + W(s_1^{q_1}, q_1)\left\{-C_1(r_2^{q_1*})\right\} & \\ + \mu_1 - \mu_2 & = 0. \end{aligned}$$

There are four cases: 1-$\{\mu_1 > 0, \mu_2 > 0\}$, 2-$\{\mu_1 > 0, \mu_2 = 0\}$, 3-$\{\mu_1 = 0, \mu_2 > 0\}$, 4-$\{\mu_1 = 0, \mu_2 = 0\}$. I examine each of these in turn.

A.2.1 Case 1:

Since $\mu_1 > 0$ and $\mu_2 > 0$, $r_1 = s_1^{q_1} = 0$. This can only be true if $r_1 = 0$, in which case, this is the only possible solution. If $r_1 > 0$, this is not possible.

A.2.2 Case 2:

Here $\mu_1 > 0$ and $\mu_2 = 0$. Thus, $s_1^{q_1} = 0$. For this to be true,

$$W_1(0, q_1)\left\{1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 + r_2^{q_1*}, q_2) dq_2\right\} + \mu_1 = W(0, q_1)\left\{C_1(r_2^{q_1*})\right\}.$$

Since $\mu_1 > 0$,

$$W_1(0, q_1)\left\{1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 + r_2^{q_1*}, q_2) dq_2\right\} < W(0, q_1)\left\{C_1(r_2^{q_1*})\right\}.$$

Rearranging,

$$\frac{W_1(0, q_1)}{C_1(r_2^{q_1*})} < \frac{W(0, q_1)}{\left\{1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 + r_2^{q_1*}, q_2) dq_2\right\}}. \tag{5}$$

To see that this is not possible, first define $\bar{r}$ as the r that solves $C_1(r) = \int_Q f(q) W(r, q) dq$. This would be the r_2 chosen by the incumbent if no war chest were brought from the first election ($r_1 = s_1$). This $\bar{r}$ exists by the assumptions that $W_1(0,q) > C_1(0)$ and there exists some $\tilde{r}$ such that $W(\tilde{r},q) > C(\tilde{r})$—$\bar{r}$ will be one of those $\tilde{r}$. From the first maximization problem solved (Equation (2)), $\int_{Q_2} f(q_2)W_1(r_1+\check{r}_2^{q_1*},q_2)dq_2 = C_1(\check{r}_2^{q_1*})$, where $\check{r}_2^{q_1*}$ is the specific $r_2^{q_1*}$ that solves this equation. From the restrictions on the first and second derivatives of $W(\cdot,\cdot)$ and $C(\cdot)$, I find that $\check{r}_2^{q_1*} < \bar{r} < r_1 + \check{r}_2^{q_1*}$ (remembering that r_1 must be positive, else this case is not possible). Since there exists some $\tilde{r}$ such that $W(\tilde{r},q) > C(\tilde{r})$, there also exists an $\bar{r}$ such that $\int_Q f(q)W(\bar{r},q)dq > C(\bar{r})$. This implies that $\int_{Q_2} f(q_2)W(r_1+r_2^{q_1*},q_2)dq_2 > C(r_2^{q_1*})$. Hence the denominator on the RHS of Equation (5) is greater than 1, while the numerator on the RHS is less than 1. Thus the fraction on the RHS is less than 1. From the Inada-type conditions on $W_1(s,q)$, note that the LHS is greater than 1. Thus, the inequality cannot hold, and this case is not possible.

A.2.3 Case 3:

In this case, $\mu_1 = 0$ and $\mu_2 > 0$, which implies that $s_1^{q_1} = r_1 > 0$. This yields the first order condition:

$$W_1(r_1,q_1)\left\{1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2)W(r_2^{q_1*},q_2)dq_2\right\} = W(r_1,q_1)\left\{C_1(r_2^{q_1*})\right\} + \mu_2.$$

Assuming that the solution is not at a corner, this can be made into an identity:

$$W_1(r_1,q_1)\left\{1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2)W(r_2^{q_1*},q_2)dq_2\right\} \equiv W(r_1,q_1)\left\{C_1(r_2^{q_1*})\right\} + \mu_2^*.$$

Since $s_1^{q_1} = r_1$, $\dfrac{ds_1^{q_1}}{dr_1} = 1$, which implies (from Equation (4)) $\dfrac{dr_2^{q_1*}}{dr_1} = 0$. Differentiating both sides with respect to r_1,

$$W_{11}(r_1,q_1)\left\{\begin{array}{l} 1 - C(r_2^{q_1*}) + \\ \int_{Q_2} f(q_2)W(r_2^{q_1*},q_2)dq_2 \end{array}\right\} = W_1(r_1,q_1)\left\{C_1(r_2^{q_1*})\right\} + \frac{d\mu_2^*}{dr_1}. \tag{6}$$

From the assumptions on $W(\cdot,\cdot)$ and $C(\cdot)$, I find that $\dfrac{d\mu_2^*}{dr_1} < 0$. This means that if there is no war chest, if one decreases r_1, there will still be no war chest. Conversely, if one increases r_1, there may be a war chest (as one hits the corner where $\mu_2^* = 0$, one moves into Case 4).

A.2.4 Case 4:

Here, $\mu_1 = 0$ and $\mu_2 = 0$. This implies that $r_1 > s_1^{q_1} > 0$. This yields the first-order condition:

$$\begin{aligned} &W_1(s_1^{q_1},q_1)\left\{1 - C(r_2^{q_1*}) + \textstyle\int_{Q_2} f(q_2)W(r_1 - s_1^{q_1} + r_2^{q_1*},q_2)dq_2\right\} \\ &= W(s_1^{q_1},q_1)\left\{C_1(r_2^{q_1*})\right\}. \end{aligned} \tag{7}$$

Since the term in braces on the LHS is greater than 1, and $W(s_1^{q_1},q_1) < 1$, $C_1(r_2^{q_1*}) > W_1(s_1^{q_1},q_1)$. Substituting in the solution value of $s_1^{q_1*}$,

$$\begin{aligned} &W_1(s_1^{q_1*},q_1)\left\{1 - C(r_2^{q_1*}) + \textstyle\int_{Q_2} f(q_2)W(r_1 - s_1^{q_1*} + r_2^{q_1*},q_2)dq_2\right\} \\ &\equiv W(s_1^{q_1*},q_1)\left\{C_1(r_2^{q_1*})\right\}. \end{aligned} \tag{8}$$

Differentiating both sides with respect to r_1, and substituting in Equation (3) to simplify,

$$\begin{aligned} &W_{11}(s_1^{q_1*},q_1)\left\{1 - C(r_2^{q_1*}) + \textstyle\int_{Q_2} f(q_2)W(r_1 - s_1^{q_1*} + r_2^{q_1*},q_2)dq_2\right\}\frac{ds_1^{q_1*}}{dr_1} \\ &= W_1(s_1^{q_1*},q_1)C_1(r_2^{q_1*})\left[2\frac{ds_1^{q_1*}}{dr_1} - 1\right] + W(s_1^{q_1*},q_1)C_{11}(r_2^{q_1*})\frac{dr_2^{q_1*}}{dr_1}. \end{aligned} \tag{9}$$

From Equation (4), if $\frac{dr_2^{q_1*}}{dr_1} > (=) \ 0$, then $\frac{ds_1^{q_1*}}{dr_1} > (=) \ 1$. This yields a contradiction. Thus, $\frac{dr_2^{q_1*}}{dr_1} < 0$, which means that $\frac{ds_1^{q_1*}}{dr_1} < 1$. If $\frac{ds_1^{q_1*}}{dr_1} \leq 0$, this yields another contradiction. Thus, $0 < \frac{ds_1^{q_1*}}{dr_1} < 1$.

A.3 Solving for money raised in the first election

Now solve the grand maximization problem:

$$\max_{r_1 \geq 0} -C(r_1) + \int_{Q_1} f(q_1) W(s_1^{q_1*}, q_1) \left[1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2 \right] dq_1$$

where $0 < s_1^{q_1*} \leq r_1$. Using the method of Kuhn-Tucker, set up the Lagrangian:

$$\begin{aligned} L(r_1, \nu) = \quad & -C(r_1) + \textstyle\int_{Q_1} f(q_1) W(s_1^{q_1*}, q_1) \left[\begin{array}{l} 1 - C(r_2^{q_1*}) + \\ \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2 \end{array} \right] dq_1 \\ & + \nu r_1. \end{aligned}$$

The first-order conditions are (again, substituting with Equation (3))

$$\begin{aligned} \frac{dL}{dr_1} = \quad & -C_1(r_1) \\ & + \int_{Q_1} f(q_1) \left\{ \begin{array}{l} W_1(s_1^{q_1*}, q_1) \frac{ds_1^{q_1*}}{dr_1} \left[\begin{array}{l} 1 - C(r_2^{q_1*}) + \\ \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2 \end{array} \right] \\ + W(s_1^{q_1*}, q_1) C_1(r_2^{q_1*}) \left(1 - \frac{ds_1^{q_1*}}{dr_1} \right) \end{array} \right\} dq_1 \\ & + \nu \\ = \quad & 0 \end{aligned}$$

$$\nu r_1 = 0, \quad \nu \geq 0, \quad r_1 \geq 0.$$

There are two cases: 1-$\{\nu > 0\}$and 2-$\{\nu = 0\}$.

A.3.1 Case 1:

Since $\nu > 0$, then $r_1 = 0$, and thus $s_1^{q_1*} = 0$. For this to be a maximum

$$\begin{aligned} & -C(0) + \textstyle\int_{Q_1} f(q_1) W(0, q_1) \left[1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_2^{q_1*}, q_2) dq_2 \right] dq_1 \geq \\ & -C(r_1) + \textstyle\int_{Q_1} f(q_1) W(r_1, q_1) \left[1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_2^{q_1*}, q_2) dq_2 \right] dq_1 \end{aligned}$$

for all $r_1 = s_1^{q_1*}$.[2] Since the term in brackets is greater than 1, if $r_1 = \varepsilon$, the inequality will not hold as a result of the Inada-type conditions on $W(\cdot, \cdot)$ and $C(\cdot)$. This also rules out the possibility of Case 1 in the second maximization problem (in Section A.2.1).

A.3.2 Case 2:

This is the only possibility that allows $r_1 > 0$. Thus, $r_1^* > 0$.

[2] It must also be greater for all $r_1 > s_1^{q*}$, but the inequality stated will be sufficient to demonstrate the contradiction.

A.4 Solution maximum

I show that this problem satisfies the conditions of the Theorem of Kuhn and Tucker under Convexity (Sundaram 1996, Theorem 7.16), and thus, that r_1^* is the maximum. Let

$$g(r_1) = -C(r_1) + \int_{Q_1} f(q_1) W(s_1^{q_1*}, q_1) \left[1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2\right] dq_1$$

so that the incumbent maximizes $g(r_1)$. If $[g'(\hat{r}_1) - g'(r_1)](\hat{r}_1 - r_1) \leq 0$ for all $\hat{r}_1, r_1 \geq 0$, then g is concave. First, separate Q_1 into two regions: a high-q region ($\overline{Q}$) in which case there will be no war chest (Case 3, in Section A.2.3), and a low-q region ($\underline{Q}$) where there will be a war chest (Case 4, in Section A.2.4). Using the first derivative from the Lagrangian on r_1,

$$\begin{aligned} g'(r_1) = \quad & -C_1(r_1) \\ & + \int_{\overline{Q}_1} f(q_1) \left\{ \begin{array}{l} W_1(s_1^{q_1*}, q_1) \dfrac{ds_1^{q_1*}}{dr_1} \left[\begin{array}{l} 1 - C(r_2^{q_1*}) + \\ \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2 \end{array} \right] \\ + W(s_1^{q_1*}, q_1) C_1(r_2^{q_1*}) \left(1 - \dfrac{ds_1^{q_1*}}{dr_1}\right) \end{array} \right\} dq_1 \\ & + \int_{\underline{Q}_1} f(q_1) \left\{ \begin{array}{l} W_1(s_1^{q_1*}, q_1) \dfrac{ds_1^{q_1*}}{dr_1} \left[\begin{array}{l} 1 - C(r_2^{q_1*}) + \\ \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2 \end{array} \right] \\ + W(s_1^{q_1*}, q_1) C_1(r_2^{q_1*}) \left(1 - \dfrac{ds_1^{q_1*}}{dr_1}\right) \end{array} \right\} dq_1. \end{aligned}$$

In $\overline{Q}$, $r_1 = s_1^{q_1*}$, and thus $\dfrac{ds_1^{q_1*}}{dr_1} = 1$. In $\underline{Q}$, $r_1 > s_1^{q_1*}$, which means that Equation (8) holds. Substituting these in, we obtain

$$\begin{aligned} g'(r_1) = \quad & -C_1(r_1) \\ & + \int_{\overline{Q}_1} f(q_1) \left\{ W_1(s_1^{q_1*}, q_1) \left[1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2\right] \right\} dq_1 \\ & + \int_{\underline{Q}_1} f(q_1) \left\{ \begin{array}{l} W_1(s_1^{q_1*}, q_1) \dfrac{ds_1^{q_1*}}{dr_1} \left[\begin{array}{l} 1 - C(r_2^{q_1*}) + \\ \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2 \end{array} \right] + \\ W_1(s_1^{q_1*}, q_1) \left[\begin{array}{l} 1 - C(r_2^{q_1*}) + \\ \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2 \end{array} \right] \left(1 - \dfrac{ds_1^{q_1*}}{dr_1}\right) \end{array} \right\} dq_1. \end{aligned}$$

We finally obtain

$$g'(r_1) = -C_1(r_1) + \int_{Q_1} f(q_1) \left\{ W_1(s_1^{q_1*}, q_1) \left[1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2\right] \right\} dq_1.$$

Let

$$g'(\hat{r}_1) = -C_1(\hat{r}_1) + \int_{Q_1} f(q_1) \left\{ W_1(\hat{s}_1^{q_1*}, q_1) \left[1 - C(\hat{r}_2^{q_1*}) + \int_{Q_2} f(q_2) W(\hat{r}_1 - \hat{s}_1^{q_1*} + \hat{r}_2^{q_1*}, q_2) dq_2\right] \right\} dq_1.$$

Without loss of generality, let $\hat{r}_1 > r_1$. (A) Since $C_{11}(r) > 0$, $-C_1(\hat{r}_1) < -C_1(r_1)$. (B) Since $W_{11}(s, \cdot) < 0$ and $\dfrac{ds_1^{q_1*}}{dr_1} > 0$, $W_1(\hat{s}_1^{q_1*}, q_1) < W_1(s_1^{q_1*}, q_1)$. (C) Since $C_1(r) > 0$ and $\dfrac{dr_2^{q_1*}}{dr_1} \leq 0$, $C(\hat{r}_2^{q_1*}) \leq C(r_2^{q_1*})$. (D) Since $W_1(s, \cdot) > 0$ and $\dfrac{d}{dr_1}(r_1 - s_1^{q_1*} + r_2^{q_1*}) \geq 0$ (see Equation 4), $W(\hat{r}_1 - \hat{s}_1^{q_1*} + \hat{r}_2^{q_1*}, q_2) \geq W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2)$.

(E) From (B), (C), and (D),

$$\begin{aligned}&\int_{Q_1} f(q_1)\left\{W_1(\hat{s}_1^{q_1*}, q_1)\left[1 - C(\hat{r}_2^{q_1*}) + \int_{Q_2} f(q_2) W(\hat{r}_1 - \hat{s}_1^{q_1*} + \hat{r}_2^{q_1*}, q_2) dq_2\right]\right\} dq_1 \\ \leq\ &\int_{Q_1} f(q_1)\left\{W_1(s_1^{q_1*}, q_1)\left[1 - C(r_2^{q_1*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{q_1*} + r_2^{q_1*}, q_2) dq_2\right]\right\} dq_1.\end{aligned}$$

From (A) and (E), we obtain $g'(\hat{r}_1) - g'(r_1) < 0$. Since $\hat{r}_1 > r_1$, we have $[g'(\hat{r}_1) - g'(r_1)](\hat{r}_1 - r_1) \leq 0$ for all $\hat{r}_1, r_1 \geq 0$. The constraint $h(r_1) = r_1 \geq 0$ is concave, and has a point $h(\overline{r}_1) > 0$ (specifically, any $r_1 > 0$), thus fulfilling Slater's condition. Since the other Kuhn-Tucker first-order conditions hold, r_1^* is the maximum.□

A.5 Relationship between challenger quality and war chest

Now I am left to show when a war chest may exist, i.e., when $r_1^* > s_1^{q*}$, and the relationship between the size of a war chest and challenger quality. To show this, first solve for r_1 if the incumbent knew for certain that she would face a very low quality challenger in the first election. Call this r_1^L. I now have the maximization problem:

$$\max_{r_1^L \geq 0} -C(r_1^L) + W(s_1^{L*}, q_1)\left[1 - C(r_2^{L*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{L*} + r_2^{L*}, q_2) dq_2\right].$$

Through the Kuhn-Tucker method, obtain the Lagrangian:

$$L(r_1^L, \nu^L) = -C(r_1^L) + W(s_1^{L*}, q_1)\left[1 - C(r_2^{L*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{L*} + r_2^{L*}, q_2) dq_2\right] + \nu^L r_1^L.$$

The first-order conditions are

$$\begin{aligned}\frac{dL}{dr_1^L} =\ & -C_1(r_1^L) \\ & + W_1(s_1^{L*}, L)\left[\begin{array}{c} 1 - C(r_2^{L*}) + \\ \int_{Q_2} f(q_2) W(r_1 - s_1^{L*} + r_2^{L*}, q_2) dq_2 \end{array}\right]\frac{ds_1^{L*}}{dr_1^L} \\ & + W(s_1^{L*}, L) C_1(r_2^{L*})\left[1 - \frac{ds_1^{L*}}{dr_1^L}\right] \\ & + \nu^L \\ =\ & 0\end{aligned} \tag{10}$$

$$\nu^L r_1^L = 0, \quad \nu^L \geq 0, \quad r_1^L \geq 0.$$

As before, I can rule out $r_1^L = 0$, which implies $\nu^L = 0$. If (as in Case 3 of Section A.2.3) $r_1^L = s_1^{L*}$, then $\frac{ds_1^{L*}}{dr_1^L} = 1$, and Equation (10) reduces to

$$C_1(r_1^L) = W_1(s_1^{L*}, L)\left[1 - C(r_2^{L*}) + \int_{Q_2} f(q_2) W(r_1 - s_1^{L*} + r_2^{L*}, q_2) dq_2\right]. \tag{11}$$

This makes sense in that the marginal cost in this election must be equal to the marginal benefit of winning this election *plus* the expected utility of the next election. And if (as in Case 4 of Section A.2.4) $r_1^L > s_1^{L*}$, then (from Equation (8)):

$$W_1(s_1^{L*}, L)\left[\begin{array}{c} 1 - C(r_2^{L*}) + \\ \int_{Q_2} f(q_2) W(r_1 - s_1^{L*} + r_2^{L*}, q_2) dq_2 \end{array}\right] = W(s_1^{L*}, L) C_1(r_2^{L*}). \tag{12}$$

Substituting Equation (12) into Equation (10), I obtain Equation (11) again. If $r_1^L > s_1^{L*}$, substitute Equation (11) into Equation (12) to get

$$C_1(r_1^L) = W(s_1^{L*}, L)C_1(r_2^{L*}). \tag{13}$$

Since $W(s_1^{L*}, L) < 1$, this requires that $C_1(r_2^{L*}) > C_1(r_1^L)$, and thus, $r_2^{L*} > r_1^L$ (if $r_1^L > s_1^{L*}$).

Going through the same process to find r_1^H for a very high quality challenger and its accompanying conditions, if $r_1^H > s_1^{H*}$, then

$$C_1(r_1^H) = W(s_1^{H*}, H)C_1(r_2^{H*})$$

and $r_2^{H*} > r_1^H$. Since by assumption, $r_L < r_H$ (the monies raised and spent if there were only one election and the incumbent knew the quality of her challenger was low or high, respectively), then the result (from Equation (7)) that $C_1(r_2^{H*}) > W_1(s_1^{H*}, H)$ yields $s_1^{H*} > r_2^{H*}$. But this means that $s_1^{H*} > r_2^{H*} > r_1^H$, and since $s_1^{H*} \leq r_1^H$, this yields a contradiction. Therefore, $r_1^H = s_1^{H*}$: An incumbent would never have a war chest if she knew that she was facing a very high quality challenger in the first election.

Now I show that $r_1^L < r_1^H$. There are two cases: 1-$\{r_1^L = s_1^{L*}\}$and 2-$\{r_1^L > s_1^{L*}\}$ (knowing that $r_1^H = s_1^{H*}$).

A.5.1 Case 1:

I wish to show that $r_1^L < r_1^H$ when $r_1^L = s_1^{L*}$. Suppose not. Then $r_1^L \geq r_1^H$, which implies that $C_1(r_1^L) \geq C_1(r_1^H)$. Substituting in the reduced first-order conditions from Equation (11),

$$W_1(r_1^L, L)\left[1 - C(r_2^{L*}) + \int_{Q_2} f(q_2)W(r_2^{L*}, q_2)dq_2\right] \geq$$
$$W_1(r_1^H, H)\left[1 - C(r_2^{H*}) + \int_{Q_2} f(q_2)W(r_2^{H*}, q_2)dq_2\right]$$

where $r_2^{L*} = r_2^{H*} = \bar{r}$ of Section A.2.2. Canceling out, I obtain

$$W_L'(r_1^L) \geq W_H'(r_1^H).$$

This yields a contradiction to the result that $r_L < r_H$.[3] Thus, $r_1^L < r_1^H$ (if there is no war chest).

A.5.2 Case 2:

I wish to show that $r_1^L < r_1^H$ when $r_1^L > s_1^{L*}$. Since $r_1^H = s_1^{H*}$, the incumbent brings no war chest with her into the second election cycle (if she draws a high quality challenger in the first election), and $r_2^{H*} = \bar{r}$. From Equation (11), $C_1(r_1^H) > W_1(r_1^H, H)$. This implies that $r_1^H > \bar{r}$. Since $r_1^L > s_1^{L*}$, $r_2^{L*} < \bar{r}$. From Equation (13), $r_2^{L*} > r_1^L$, and thus $r_1^L < \bar{r}$. Therefore $r_1^L < r_1^H$.

Since there is a distribution of q, by the concavity of the overall utility function, one obtains $r_1^L < r_1^* < r_1^H$. Now I examine when to expect a war chest. If the incumbent knew she was running against a very high quality challenger, she would raise r_1^H (and spend $s_1^{H*} = r_1^H$). From the point of view of the model, she is in Case 3, and thus, $\mu_2 > 0$. Since $\frac{d\mu_2^*}{dr_1} < 0$ (from Equation (6)), as one decreases r_1 from r_1^H to r_1^*, μ_2^* increases, and thus, there is no war chest if an incumbent found out she was running against a high quality challenger *after* she raised the money. If the incumbent knew she was running against a low quality challenger, she would raise r_1^L. She may save some or none of it in this case. If she had no war chest in this case, since $\frac{d\mu_2^*}{dr_1} < 0$, as one increases r_1 from r_1^L to r_1^*, μ_2^* decreases—perhaps reaching 0, in which case there

[3] Or, more specifically, this yields a contradition to the shapes that $W(\cdot, L)$ and $W(\cdot, H)$ must take to maintain the result that $r_L < r_H$.

would be a war chest. If the incumbent had a war chest at r_1^L, then since $0 < \frac{ds_1^{L*}}{dr_1^L} < 1$ (from Equation (9)), as one increased r_1 from r_1^L to r_1^*, the incumbent would keep a larger and larger war chest. Therefore, $\frac{d}{dq_1}(r_1^* - s_1^{q_1*}) \leq 0$. In other words, the size of an incumbent's war chest (weakly) decreases as challenger quality increases.□

A.6 Relationship between potential challenger quality and money raised

Recall that according to Equation 2,

$$\int_{Q_2} f(q_2)W_1(r_1 - s_1^{q_1} + r_2^{q_1*}, q_2)dq_2 \equiv C_1(r_2^{q_1*}).$$

By assumption, the r that solves $W_1(r, q) = C_1(r)$ is greater than the $\hat{r}$ that solves $W_1(\hat{r}, \hat{q}) = C_1(\hat{r})$ for $q > \hat{q}$. By extension, as the average q_2 increases (through $f(q_2)$ or Q_2), r_2 will also increase. A similar argument follows for q_1.□

A.7 Relationship between money spent previously and money raised

From the reasoning in Section A.1.2, when there is a war chest, $0 < \frac{\partial r_2^{q_1*}}{\partial s_1^{q_1}} < 1$. If there is not a war chest, then $\frac{\partial r_2^{q_1*}}{\partial s_1^{q_1}} = 0$. Combining these two cases together,

$$0 \leq \frac{\partial r_2^{q_1*}}{\partial s_1^{q_1}} < 1.$$

Thus, the more the incumbent spends in the first election, the more she will raise in the second election.□

A.8 Relationship between war chests and money raised

Using Equation 2,

$$\int_{Q_2} f(q_2)W_1(r_1 - s_1^{q_1} + r_2^{q_1*}, q_2)dq_2 \equiv C_1(r_2^{q_1*}).$$

Partially differentiating both sides with respect to the war chest $(r_1 - s_1^{q_1})$, I obtain

$$\left\{\int_{Q_2} f(q_2)W_{11}(r_1 - s_1^{q_1} + r_2^{q_1*}, q_2)dq_2 \left[1 + \frac{\partial r_2^{q_1*}}{\partial(r_1 - s_1^{q_1})}\right]\right\} \equiv C_{11}(r_2^{q_1*})\frac{\partial r_2^{q_1*}}{\partial(r_1 - s_1^{q_1})}.$$

Since $W_{11}(s, q) < 0$ and $C_{11}(r) > 0$, I find that $-1 < \frac{\partial r_2^{q_1*}}{\partial(r_1 - s_1^{q_1})} < 0$. Thus, as the war chest increases, the money raised by the incumbent in the second period decreases, but not as fast as the war chest increases.□

A.9 Relationship between money raised and money spent

As shown in Sections A.2.3 and A.2.4, $\frac{ds_1^{q_1*}}{dr_1} = 1$ (Case 3), or $0 < \frac{ds_1^{q_1*}}{dr_1} < 1$ (Case 4). Combining these two cases together,

$$0 < \frac{ds_1^{q_1*}}{dr_1} \leq 1.$$

Thus, as the incumbent raises more money, she will spend more money (in the first election), but she will not spend more money than she raises.□

Table A: Summary Statistics

Variable	**Mean**	**Median**	**Std. Dev.**	**Minimum**	**Maximum**
War Chest	$182,000	106,000	260,000	–32,000	5,248,000
Fund-raising	$597,000	518,000	405,000	8,000	6,495,000
Spending	$539,000	451,000	404,000	8,000	6,541,000
Number of Districts	18.6	14	13.8	1	52
Tenure	9.99	8	7.89	2	51
Party Advantage	5.99	5	11.8	–28.0	46.1
Competitive Primary	0.30	0	0.53	0	3
Scandal	0.07	0	0.26	0	1
Grandfather	0.33	0	0.47	0	1
Redistricting	0.24	0	0.43	0	1
Democrat	0.57	1	0.50	0	1
South	0.26	0	0.44	0	1

Notes: 1982-1998 U.S. House races with incumbents, excluding unusual cases.
All dollar amounts in 1998 dollars.

B Empirical Results

Table A contains summary statistics for the variables used.

Table B contains the control variables of year-specific Democrat, year-specific South, Year, and constant that were included in the statistical specifications of the Propositions, but not reported in Tables. The first set of columns of Table B contains the control variables for the test of Proposition 1 not reported in Table 2. The second set of columns of Table B contains the control variables for the tests of Propositions 2, 3, and 4 not reported in Table 3. The last set of columns of Table B contains the control variables for the tests of Proposition 5 not reported in Table 4.

Table B: Control Variables for Tests of Propositions (Tables 2, 3, and 4)

	***log* War Chest**		***log* Funds Raised**		***log* Funds Spent**	
Variable	Coefficient	(t-ratio)	Coefficient	(t-ratio)	Coefficient	(t-ratio)
Democrat in 1998	0.927	(1.68)	0.533*	(2.85)	–0.99	(–0.89)
Democrat in 1996	0.971	(1.76)	0.505*	(2.80)	–0.018	(–0.17)
Democrat in 1994	0.890	(1.63)	0.777*	(4.23)	0.026	(0.24)
Democrat in 1992	1.298*	(2.35)	0.676*	(3.67)	–0.051	(–0.46)
Democrat in 1990	1.513*	(2.65)	0.521*	(2.65)	–0.043	(–0.36)
Democrat in 1988	1.418*	(2.48)	0.389	(1.95)	–0.087	(–0.74)
Democrat in 1986	1.322*	(2.30)	0.386	(1.92)	–0.036	(–0.30)
Democrat in 1984	0.925	(1.61)	0.291	(1.43)	–0.026	(–0.22)
Democrat in 1982	0.990	(1.72)	-		–0.046	(–0.39)
South in 1998	–0.283	(–1.10)	†		0.006	(0.15)
South in 1996	–0.069	(–0.25)	0.071	(0.92)	†	
South in 1994	–0.176	(–0.69)	0.131	(1.42)	–0.009	(–0.21)
South in 1992	–0.118	(–0.48)	–0.031	(–0.32)	–0.036	(–0.64)
South in 1990	–0.396	(–1.67)	–0.003	(–0.03)	–0.016	(–0.26)
South in 1988	–0.238	(–1.10)	0.023	(0.22)	–0.032	(–0.57)
South in 1986	–0.034	(–0.16)	0.112	(1.04)	–0.104	(–1.75)
South in 1984	–0.120	(-0.58)	0.008	(0.07)	–0.126*	(–2.17)
South in 1982	†		-		–0.116	(–1.77)
1996	–0.924*	(–6.76)	0.569	(1.27)	0.110*	(3.26)
1994	–1.093*	(–5.27)	0.661	(0.74)	0.103*	(2.31)
1992	–1.292*	(–3.97)	1.150	(0.85)	0.192*	(3.00)
1990	–1.010*	(–2.66)	1.532	(0.86)	0.049	(0.66)
1988	–0.967*	(–2.09)	2.028	(0.91)	0.050	(0.58)
1986	–1.086*	(–1.98)	2.353	(0.88)	0.010	(0.10)
1984	–1.182	(–1.88)	2.679	(0.86)	–0.019	(–0.17)
1982	–1.785*	(–2.49)	-		0.047	(0.037)

Notes: $^*p < 0.05$ (two-tailed test)

†Dropped due to collinearity with fixed effects.

References

[1] Sundaram, Rangarajan K. 1996. *A First Course in Optimization Theory*. New York: Cambridge University Press.