Appendix to "When Are War Chests Informative?"

Jay Goodliffe

August 2005

In this appendix, I sign the comparative statics—such as whether fundraising increases in incumbent strength—for the incumbent holding challenger quality constant, i.e. when challenger entry is not strategic. I then give detailed proofs for the equilibria presented in the main text.

A Proofs of Comparative Statics

In this section, I present the solution to the incumbent's maximization problem holding (probability of) challenger quality constant. The method is to work backwards through each decision point of the incumbent solving each maximization problem with the Kuhn-Tucker method. I then show how war chests and other variables change with incumbent strength through the implicit function theorem.

Because $W_1(s, I) > 0$,[1] the incumbent will spend all that he has in the last election: $s_2 = r_1 - s_1 + r_2$ (recall that subscripts denote the first or second election). Since the probabilities of drawing a high quality challenger are held constant here (that is, they are known by the incumbent), the incumbent's utility maximization problem is a nested maximization problem:

$$\begin{aligned}
&\max_{r_1 \geq 0} -C(r_1) \\
&+\eta_1 \left\{ \max_{0 \leq s_1^H \leq r_1} W^H(s_1^H, I) \left\langle 1 + \max_{r_2^H \geq 0} \left[\begin{array}{c} -C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right] \right\rangle \right\} \\
&+(1-\eta_1) \left\{ \max_{0 \leq s_1^L \leq r_1} W^L(s_1^L, I) \left\langle 1 + \max_{r_2^L \geq 0} \left[\begin{array}{c} -C(r_2^L) + \eta_2 W^H(r_1 - s_1^L + r_2^L, I) \\ +(1-\eta_2) W^L(r_1 - s_1^L + r_2^L, I) \end{array} \right] \right\rangle \right\}
\end{aligned}$$

I solve the inner maximization problems and work outwards (to the overall maximization problem).

A.1 Solving for money raised in the second election

First solve:

$$\max_{r_2^j \geq 0} \left[-C(r_2^j) + \eta_2 W^H(r_1 - s_1^j + r_2^j, I) + (1-\eta_2) W^L(r_1 - s_1^j + r_2^j, I) \right]$$

(The method is the same whether a low or high quality challenger ran in the first election, *i.e.*, $j = L$ or H.) For this problem, $r_1 - s_1^j$ (the war chest) is a constant. Using the method of Kuhn-Tucker to solve this problem, I first set up the Lagrangian:

$$L(r_2^j, \lambda) = -C(r_2^j) + \eta_2 W^H(r_1 - s_1^j + r_2^j, I) + (1-\eta_2) W^L(r_1 - s_1^j + r_2^j, I) + \lambda r_2^j.$$

The first-order conditions are

$$\frac{\partial L}{\partial r_2^j} = -C_1(r_2^j) + \eta_2 W_1^H(r_1 - s_1^j + r_2^j, I) + (1-\eta_2) W_1^L(r_1 - s_1^j + r_2^j, I) + \lambda = 0$$
$$\lambda \geq 0, \quad r_2^j \geq 0, \quad \lambda r_2^j = 0.$$

There are two cases: 1-$\{\lambda > 0,\ r_2^j = 0\}$and 2-$\{\lambda = 0,\ r_2^j > 0\}$. I examine the cases separately.

[1] Following convention, a subscript n on a function denotes the partial derivative of that function with respect to the nth argument of the function. Thus, $W_1(s, q) = \dfrac{\partial W(s, q)}{\partial s}$.

A.1.1 Case 1:

Here $r_2^j = 0$. Therefore,

$$\eta_2 W_1^H(r_1 - s_1^j, I) + (1-\eta_2) W_1^L(r_1 - s_1^j, I) < C_1(0).$$

Since $C_1(0) = 0$ and $W_1^j(\cdot,\cdot) > 0$ by assumption, this cannot be true.

A.1.2 Case 2:

Here $\lambda = 0$. For a given war chest $(r_1 - s_1^j)$, find the $r_2^j > 0$ that solves

$$\eta_2 W_1^H(r_1 - s_1^j + r_2^j, I) + (1-\eta_2) W_1^L(r_1 - s_1^j + r_2^j, I) = C_1(r_2^j). \tag{1}$$

For future use I derive some comparative statics. Substituting the r_2^{j*} that solves Equation (1), I obtain

$$\eta_2 W_1^H(r_1 - s_1^j + r_2^{j*}, I) + (1-\eta_2) W_1^L(r_1 - s_1^j + r_2^{j*}, I) \equiv C_1(r_2^{j*}). \tag{2}$$

Partially differentiating both sides with respect to s_1^j, I obtain

$$\left\{\eta_2 W_{11}^H(r_1 - s_1^j + r_2^{j*}, I) + (1-\eta_2) W_{11}^L(r_1 - s_1^j + r_2^{j*}, I)\right\}\left\{-1 + \frac{\partial r_2^{j*}}{\partial s_1^j}\right\} = C_{11}(r_2^{j*})\frac{\partial r_2^{j*}}{\partial s_1^j}.$$

Since $W_{11}^j(\cdot,\cdot) < 0$ and $C_{11}(\cdot) > 0$, I find that $0 < \dfrac{\partial r_2^{j*}}{\partial s_1^j} < 1$. Looking further ahead, I substitute the r_2^{j*} that solves Equation (1) with the equilibrium s_1^{j*}:

$$\eta_2 W_1^H(r_1 - s_1^{j*} + r_2^{j*}, I) + (1-\eta_2) W_1^L(r_1 - s_1^{j*} + r_2^{j*}, I) \equiv C_1(r_2^{j*}). \tag{3}$$

Differentiating both sides with respect to r_1, I obtain

$$\left\{\begin{array}{c} \eta_2 W_{11}^H(r_1 - s_1^{j*} + r_2^{j*}, I) + \\ (1-\eta_2) W_{11}^L(r_1 - s_1^{j*} + r_2^{j*}, I) \end{array}\right\}\left\{1 - \frac{ds_1^{j*}}{dr_1} + \frac{dr_2^{j*}}{dr_1}\right\} = C_{11}(r_2^{j*})\frac{dr_2^{j*}}{dr_1}.$$

Since $W_{11}(\cdot,\cdot) < 0$ and $C_{11}(\cdot) > 0$, I find

$$\begin{array}{lcl} \dfrac{dr_2^{j*}}{dr_1} > 0 & \Leftrightarrow & \dfrac{ds_1^{j*}}{dr_1} > 1 + \dfrac{dr_2^{j*}}{dr_1} > 1 \\ \dfrac{dr_2^{j*}}{dr_1} = 0 & \Leftrightarrow & \dfrac{ds_1^{j*}}{dr_1} = 1 \\ \dfrac{dr_2^{j*}}{dr_1} < 0 & \Leftrightarrow & \dfrac{ds_1^{j*}}{dr_1} < 1 + \dfrac{dr_2^{j*}}{dr_1} < 1. \end{array} \tag{4}$$

A.2 Solving for the money spent in the first election

Now, solve the next maximization problem—finding s_1^j:

$$\max_{0 \le s_1^j \le r_1} W^j(s_1^j, I)\left[1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) + (1-\eta_2) W^L(r_1 - s_1^j + r_2^{j*}, I)\right].$$

(The maximization problem is similar for s_1^L and s_1^H). In this problem, r_1 is treated as a constant. Solve again through the method of Kuhn-Tucker by first setting up the Lagrangian:

$$\begin{aligned} L(s_1^j, \mu_1, \mu_2) = \; & W^H(s_1^j, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^j + r_2^{j*}, I) \end{array}\right\} \\ & + \mu_1 s_1^j + \mu_2(r_1 - s_1^j). \end{aligned}$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial s_1^j} &= W_1^j(s_1^j, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^j + r_2^{j*}, I)\end{array}\right\} \\
&\quad + W^j(s_1^j, I)\left\{\begin{array}{l} -C_1(r_2^{j*})\dfrac{\partial r_2^{j*}}{\partial s_1^j} \\ + \left[\begin{array}{c} \eta_2 W_1^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W_1^L(r_1 - s_1^j + r_2^{j*}, I)\end{array}\right]\left[-1 + \dfrac{\partial r_2^{j*}}{\partial s_1^j}\right]\end{array}\right\} \\
&\quad + \mu_1 - \mu_2 \\
&= 0
\end{aligned}$$

$$\mu_1 s_1^j = 0, \quad \mu_1 \geq 0, \quad s_1^j \geq 0$$
$$\mu_2(r_1 - s_1^j) = 0, \quad \mu_2 \geq 0, \quad r_1 \geq s_1^j.$$

Using Equation (2), substitute and simplify, and obtain

$$W_1^j(s_1^j, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^j + r_2^{j*}, I)\end{array}\right\} + W^j(s_1^j, I)\left\{-C_1(r_2^{j*})\right\} + \mu_1 - \mu_2 = 0.$$

There are four cases: 1-$\{\mu_1 > 0, \mu_2 > 0\}$, 2-$\{\mu_1 > 0, \mu_2 = 0\}$, 3-$\{\mu_1 = 0, \mu_2 > 0\}$, 4-$\{\mu_1 = 0, \mu_2 = 0\}$. I examine each of these in turn.

A.2.1 Case 1:

Since $\mu_1 > 0$ and $\mu_2 > 0$, $r_1 = s_1^j = 0$. This can only be true if $r_1 = 0$, in which case, this is the only possible solution. If $r_1 > 0$, this is not possible.

A.2.2 Case 2:

Here $\mu_1 > 0$ and $\mu_2 = 0$. Thus, $s_1^j = 0$. For this to be true,

$$W_1^j(0, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 + r_2^{j*}, I)\end{array}\right\} + \mu_1 = W^j(0, I)\left\{C_1(r_2^{j*})\right\}.$$

Since $\mu_1 > 0$,

$$W_1^j(0, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 + r_2^{j*}, I)\end{array}\right\} < W^j(0, I)\left\{C_1(r_2^{j*})\right\}.$$

Rearranging,

$$\frac{W_1^j(0, I)}{C_1(r_2^{j*})} < \frac{W^j(0, I)}{\left\{1 - C(r_2^{j*}) + \eta_2 W^H(r_1 + r_2^{j*}, I) + (1-\eta_2)W^L(r_1 + r_2^{j*}, I)\right\}}. \tag{5}$$

To see that this is not possible, first define $\bar{r}$ as the r that solves $C_1(r) = pW_1^H(r, I) + (1-p)W_1^L(r, I)$.[2] This would be the r_2 chosen by the incumbent if no war chest were brought from the first election ($r_1 = s_1$). This $\bar{r}$ exists by the assumptions that $W_1^j(0, I) > C_1(0)$ and there exists some $\tilde{r}$ such that $W^j(\tilde{r}, I) > C(\tilde{r})$—$\bar{r}$ will be one of those $\tilde{r}$. From the first maximization problem solved (Equation (2)), $\eta_2 W_1^H(r_1 + r_2^{j*}, I) + (1-\eta_2)W_1^L(r_1 + r_2^{j*}, I) = C_1(\breve{r}_2^{j*})$, where $\breve{r}_2^{j*}$ is the specific r_2^{j*} that solves this equation. From the restrictions on the first and second derivatives of $W^j(\cdot,\cdot)$ and $C(\cdot)$, I find that $\breve{r}_2^{j*} < \bar{r} < r_1 + \breve{r}_2^{j*}$ (remembering that r_1 must be positive, else this case is not possible). Since there exists some $\tilde{r}$ such that $W^j(\tilde{r}, I) > C(\tilde{r})$, there also exists an $\bar{r}$ such that $pW^H(\bar{r}, I) + (1-p)W^L(\bar{r}, I) > C(\bar{r})$. This implies that $\eta_2 W^H(r_1 + r_2^{j*}, I) + (1-$

[2] Thus, $r_L < \bar{r} < r_H$.

$\eta_2)W^L(r_1 + r_2^{j*}, I) > C(r_2^{j*})$. Hence the denominator on the RHS of Equation (5) is greater than 1, while the numerator on the RHS is less than 1. Thus the fraction on the RHS is less than 1. From the Inada-type conditions on $W_1^j(\cdot,\cdot)$, note that the LHS is greater than 1. Thus, the inequality cannot hold, and this case is not possible.

A.2.3 Case 3:

In this case, $\mu_1 = 0$ and $\mu_2 > 0$, which implies that $s_1^j = r_1 > 0$. This yields the first order condition:

$$W_1^j(r_1, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_2^{j*}, I) \end{array}\right\} = W^j(r_1, I)\left\{C_1(r_2^{j*})\right\} + \mu_2.$$

Assuming that the solution is not at a corner, this can be made into an identity:

$$W_1^j(r_1, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_2^{j*}, I) \end{array}\right\} \equiv W^j(r_1, I)\left\{C_1(r_2^{j*})\right\} + \mu_2^*.$$

Since $s_1^j = r_1$, $\dfrac{ds_1^{j*}}{dr_1} = 1$, which implies (from Equation (4)) $\dfrac{dr_2^{j*}}{dr_1} = 0$. Differentiating both sides with respect to r_1,

$$W_{11}^j(r_1, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_2^{j*}, I) \end{array}\right\} = W_1^j(r_1, I)\left\{C_1(r_2^{j*})\right\} + \frac{d\mu_2^*}{dr_1}. \tag{6}$$

From the assumptions on $W^j(\cdot,\cdot)$ and $C(\cdot)$, I find that $\dfrac{d\mu_2^*}{dr_1} < 0$. This means that if there is no war chest, if one decreases r_1, there will still be no war chest. Conversely, if one increases r_1, there may be a war chest (as one hits the corner where $\mu_2^* = 0$, one moves into Case 4).

A.2.4 Case 4:

Here, $\mu_1 = 0$ and $\mu_2 = 0$. This implies that $r_1 > s_1^j > 0$. This yields the first-order condition:

$$W_1^j(s_1^j, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^j + r_2^{j*}, I) \end{array}\right\} = W^j(s_1^j, I)\left\{C_1(r_2^{j*})\right\}. \tag{7}$$

Since the term in braces on the LHS is greater than 1, and $W^j(s_1^j, I) < 1$, $C_1(r_2^{j*}) > W_1^j(s_1^j, I)$. Substituting in the solution value of s_1^j,

$$W_1^j(s_1^{j*}, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^{j*} + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^{j*} + r_2^{j*}, I) \end{array}\right\} \equiv W^j(s_1^{j*}, I)\left\{C_1(r_2^{j*})\right\}. \tag{8}$$

Differentiating both sides with respect to r_1, and substituting in Equation (3) to simplify,

$$\begin{aligned} & W_{11}^j(s_1^{j*}, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^{j*} + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^{j*} + r_2^{j*}, I) \end{array}\right\}\frac{ds_1^{j*}}{dr_1} \\ & = W_1^j(s_1^{j*}, I)C_1(r_2^{j*})\left[2\frac{ds_1^{j*}}{dr_1} - 1\right] + W^j(s_1^{j*}, I)C_{11}(r_2^{j*})\frac{dr_2^{j*}}{dr_1}. \end{aligned} \tag{9}$$

From Equation (4), if $\dfrac{dr_2^{j*}}{dr_1} > 0$, then $\dfrac{ds_1^{j*}}{dr_1} > 1$. This yields a contradiction. Thus, $\dfrac{dr_2^{j*}}{dr_1} < 0$, which means that $\dfrac{ds_1^{j*}}{dr_1} < 1$. If $\dfrac{ds_1^{j*}}{dr_1} < 0$, this yields another contradiction. Thus, $0 < \dfrac{ds_1^{j*}}{dr_1} < 1$.

A.3 Solving for money raised in the first election

Now solve the grand maximization problem:

$$\max_{r_1 \geq 0} \quad -C(r_1) \quad +\eta_1 \left\{ W^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \right\}$$
$$+(1-\eta_1) \left\{ W^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \right\}$$

where $0 < s_1^{L*} \leq r_1$ and $0 < s_1^{H*} \leq r_1$. Using the method of Kuhn-Tucker, set up the Lagrangian:

$$\begin{aligned} L(r_1, \nu) = \; & -C(r_1) \\ & +\eta_1 \left\{ W^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \right\} \\ & +(1-\eta_1) \left\{ W^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \right\} \\ & +\nu r_1. \end{aligned}$$

The first-order conditions are (again, substituting with Equation (3))

$$\begin{aligned} \frac{dL}{dr_1} = \; & -C_1(r_1) \\ & +\eta_1 \left\{ \begin{array}{l} W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \dfrac{ds_1^{H*}}{dr_1} \\ +W^H(s_1^{H*}, I) C_1(r_2^{H*}) \left[1 - \dfrac{ds_1^{H*}}{dr_1} \right] \end{array} \right\} \\ & +(1-\eta_1) \left\{ \begin{array}{l} W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \dfrac{ds_1^{L*}}{dr_1} \\ +W^L(s_1^{H*}, I) C_1(r_2^{L*}) \left[1 - \dfrac{ds_1^{L*}}{dr_1} \right] \end{array} \right\} \\ & +\nu \\ = \; & 0 \end{aligned}$$

$$\nu r_1 = 0, \quad \nu \geq 0, \quad r_1 \geq 0.$$

There are two cases: 1-$\{\nu > 0\}$and 2-$\{\nu = 0\}$.

A.3.1 Case 1:

Since $\nu > 0$, then $r_1 = 0$, and thus $s_1^{H*} = s_1^{L*} = 0$ (and $r_2^{H*} = r_2^{L*}$). For this to be a maximum

$$\begin{aligned} -C(0) + \left\{ \eta_1 W^H(0, I) + (1-\eta_1) W^L(0, I) \right\} \left[\begin{array}{r} 1 - C(r_2^*) + \eta_2 W^H(r_2^*, I) \\ +(1-\eta_2) W^L(r_2^*, I) \end{array} \right] \geq \\ -C(r_1) + \left\{ \eta_1 W^H(r_1, I) + (1-\eta_1) W^L(r_1, I) \right\} \left[\begin{array}{r} 1 - C(r_2^*) + \eta_2 W^H(r_2^*, I) \\ +(1-\eta_2) W^L(r_2^*, I) \end{array} \right] \end{aligned}$$

for all $r_1 = s_1^{H*} = s_1^{L*}$.[3] Since the term in brackets is greater than 1, if $r_1 = \varepsilon$, the inequality will not hold as a result of the Inada-type conditions on $W^j(\cdot, \cdot)$ and $C(\cdot)$. This also rules out the possibility of Case 1 in the second maximization problem (in Section A.2).

A.3.2 Case 2:

This is the only possibility that allows $r_1 > 0$. Thus, $r_1^* > 0$.

[3] It must also be greater for all $r_1 > s_1^{H*}$, etc., but the inequality stated will be sufficient to demonstrate the contradiction.

A.4 Solution Maximum

I show that this problem satisfies the conditions of the Theorem of Kuhn and Tucker under Convexity (Sundaram 1996, Theorem 7.16), and thus, that r_1^* is the maximum. Let

$$\begin{aligned} g(r_1) = \quad & -C(r_1) \quad +\eta_1 \left\{ W^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \right\} \\ & +(1-\eta_1) \left\{ W^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \right\} \end{aligned}$$

so that the incumbent maximizes $g(r_1)$. If $[g'(\hat{r}_1) - g'(r_1)](\hat{r}_1 - r_1) \leq 0$ for all $\hat{r}_1, r_1 \geq 0$, then g is concave. Using the first derivative from the Lagrangian on r_1,

$$\begin{aligned} g'(r_1) = \quad & -C_1(r_1) \\ & +\eta_1 \left\{ \begin{array}{l} W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \dfrac{ds_1^{H*}}{dr_1} \\ +W^H(s_1^{H*}, I) C_1(r_2^{H*}) \left[1 - \dfrac{ds_1^{H*}}{dr_1} \right] \end{array} \right\} \\ & +(1-\eta_1) \left\{ \begin{array}{l} W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \dfrac{ds_1^{L*}}{dr_1} \\ +W^L(s_1^{H*}, I) C_1(r_2^{L*}) \left[1 - \dfrac{ds_1^{L*}}{dr_1} \right] \end{array} \right\} \end{aligned}$$

If the first challenger is high quality, $r_1 = s_1^{H*}$, and thus $\dfrac{ds_1^{H*}}{dr_1} = 1$. If the first challenger is low quality, $r_1 \geq s_1^{L*}$. If $r_1 > s_1^{L*}$,Equation (8) holds. If $r_1 = s_1^{L*}$, then the equation is simplified. Substituting these in, we obtain

$$\begin{aligned} g'(r_1) = \quad & -C_1(r_1) \\ & +\eta_1 W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \\ & +(1-\eta_1) \left\{ \begin{array}{l} W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \dfrac{ds_1^{L*}}{dr_1} \\ +W^L(s_1^{H*}, I) C_1(r_2^{L*}) \left[1 - \dfrac{ds_1^{L*}}{dr_1} \right] \end{array} \right\}. \end{aligned}$$

We finally obtain

$$\begin{aligned} g'(r_1) = \quad & -C_1(r_1) \quad +\eta_1 W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \\ & +(1-\eta_1) W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \end{aligned} \tag{10}$$

Let

$$\begin{aligned} g'(\hat{r}_1) = \quad & -C_1(\hat{r}_1) \quad +\eta_1 W_1^H(\hat{s}_1^{H*}, I) \left[\begin{array}{r} 1 - C(\hat{r}_2^{H*}) + \eta_2 W^H(\hat{r}_1 - \hat{s}_1^{H*} + \hat{r}_2^{H*}, I) \\ +(1-\eta_2) W^L(\hat{r}_1 - \hat{s}_1^{H*} + \hat{r}_2^{H*}, I) \end{array} \right] \\ & +(1-\eta_1) W_1^L(\hat{s}_1^{L*}, I) \left[\begin{array}{r} 1 - C(\hat{r}_2^{L*}) + \eta_2 W^H(\hat{r}_1 - \hat{s}_1^{L*} + \hat{r}_2^{L*}, I) \\ +(1-\eta_2) W^L(\hat{r}_1 - \hat{s}_1^{L*} + \hat{r}_2^{L*}, I) \end{array} \right] \end{aligned}$$

Without loss of generality, let $\hat{r}_1 > r_1$. (A) Since $C_{11}(r) > 0$, $-C_1(\hat{r}_1) < -C_1(r_1)$. (B) Since $W_{11}(s, I) < 0$ and $\dfrac{ds_1^{j*}}{dr_1} > 0$, $W_1(\hat{s}_1^{j*}, I) < W_1(s_1^{j*}, I)$. (C) Since $C_1(r) > 0$ and $\dfrac{dr_2^{j*}}{dr_1} \leq 0$, $C(\hat{r}_2^{j*}) \leq C(r_2^{j*})$. (D) Since $W_1(s, I) > 0$ and $\dfrac{d}{dr_1}(r_1 - s_1^{q_1*} + r_2^{q_1*}) \geq 0$ (see Equation 4), $W(\hat{r}_1 - \hat{s}_1^{j*} + \hat{r}_2^{j*}, I) \geq W(r_1 - s_1^{j*} + r_2^{j*}, I)$.

(E) From (B), (C), and (D),

$$\left\{\begin{array}{l} \eta_1 W_1^H(\hat{s}_1^{H*}, I)\left[\begin{array}{r} 1-C(\hat{r}_2^{H*})+\eta_2 W^H(\hat{r}_1-\hat{s}_1^{H*}+\hat{r}_2^{H*}, I) \\ +(1-\eta_2)W^L(\hat{r}_1-\hat{s}_1^{H*}+\hat{r}_2^{H*}, I)\end{array}\right] \\ +(1-\eta_1)W_1^L(\hat{s}_1^{L*}, I)\left[\begin{array}{r} 1-C(\hat{r}_2^{L*})+\eta_2 W^H(\hat{r}_1-\hat{s}_1^{L*}+\hat{r}_2^{L*}, I) \\ +(1-\eta_2)W^L(\hat{r}_1-\hat{s}_1^{L*}+\hat{r}_2^{L*}, I)\end{array}\right]\end{array}\right\}$$
$$\leq\left\{\begin{array}{l} \eta_1 W_1^H(s_1^{H*}, I)\left[\begin{array}{r} 1-C(r_2^{H*})+\eta_2 W^H(r_1-s_1^{H*}+r_2^{H*}, I) \\ +(1-\eta_2)W^L(r_1-s_1^{H*}+r_2^{H*}, I)\end{array}\right] \\ +(1-\eta_1)W_1^L(s_1^{L*}, I)\left[\begin{array}{r} 1-C(r_2^{L*})+\eta_2 W^H(r_1-s_1^{L*}+r_2^{L*}, I) \\ +(1-\eta_2)W^L(r_1-s_1^{L*}+r_2^{L*}, I)\end{array}\right]\end{array}\right\}.$$

From (A) and (E), we obtain $g'(\hat{r}_1)-g'(r_1)<0$. Since $\hat{r}_1>r_1$, we have $[g'(\hat{r}_1)-g'(r_1)](\hat{r}_1-r_1)\leq 0$ for all $\hat{r}_1, r_1\geq 0$. The constraint $h(r_1)=r_1\geq 0$ is concave, and has a point $h(\overline{r}_1)>0$ (specifically, any $r_1>0$), thus fulfilling Slater's condition. Since the other Kuhn-Tucker first-order conditions hold, r_1^* is the maximum.□

A.5 The possibility of a war chest

Now I am left to show when a war chest may exist, *i.e.*, when $r_1^*>s_1^{L*}$ and/or $r_1^*>s_1^{H*}$. To show this, first solve for r_1 if the incumbent knew for certain that he would face a low quality challenger in the first election. Call this r_1^L. I now have the maximization problem:

$$\max_{r_1^L\geq 0} -C(r_1^L)+W^L(s_1^{L*}, I)\left[\begin{array}{r} 1-C(r_2^{L*})+\eta_2 W^H(r_1^L-s_1^{L*}+r_2^{H*}, I) \\ +(1-\eta_2)W^L(r_1^L-s_1^{L*}+r_2^{L*}, I)\end{array}\right].$$

Through the Kuhn-Tucker method, obtain the Lagrangian:

$$L(r_1^L, \nu^L)=-C(r_1^L)+W^L(s_1^{L*}, I)\left[\begin{array}{r} 1-C(r_2^{L*})+\eta_2 W^H(r_1^L-s_1^{L*}+r_2^{L*}, I) \\ +(1-\eta_2)W^L(r_1^L-s_1^{L*}+r_2^{L*}, I)\end{array}\right]+\nu^L r_1^L.$$

The first-order conditions are

$$\begin{aligned}\frac{dL}{dr_1^L} =\ & -C_1(r_1^L) \\ & +W_1^L(s_1^{L*}, I)\left[\begin{array}{r} 1-C(r_2^{L*})+\eta_2 W^H(r_1^L-s_1^{L*}+r_2^{L*}, I) \\ +(1-\eta_2)W^L(r_1^L-s_1^{L*}+r_2^{L*}, I)\end{array}\right]\frac{ds_1^{L*}}{dr_1^L} \\ & +W^L(s_1^{L*}, I)C_1(r_2^{L*})\left[1-\frac{ds_1^{L*}}{dr_1^L}\right] \\ & +\nu^L \\ =\ & 0\end{aligned} \tag{11}$$

$$\nu^L r_1^L=0, \quad \nu^L\geq 0, \quad r_1^L\geq 0.$$

As before, I can rule out $r_1^L=0$, which implies $\nu^L=0$. If (as in Case 3 of Section A.2 above) $r_1^L=s_1^{L*}$, then $\dfrac{ds_1^{L*}}{dr_1^L}=1$, and Equation (11) reduces to

$$C_1(r_1^L)=W_1^L(s_1^{L*}, I)\left[\begin{array}{r} 1-C(r_2^{L*})+\eta_2 W^H(r_1^L-s_1^{L*}+r_2^{L*}, I) \\ +(1-\eta_2)W^L(r_1^L-s_1^{L*}+r_2^{L*}, I)\end{array}\right]. \tag{12}$$

This makes sense in that the marginal cost in this election must be equal to the marginal benefit of winning this election *plus* the expected utility of the next election. And if (as in Case 4 of Section A.2 above) $r_1^L>s_1^{L*}$, then (from Equation (7)):

$$W_1^L(s_1^{L*}, I)\left[\begin{array}{r} 1-C(r_2^{L*})+\eta_2 W^H(r_1^L-s_1^{L*}+r_2^{H*}, I) \\ +(1-\eta_2)W^L(r_1^L-s_1^{L*}+r_2^{L*}, I)\end{array}\right]=W^L(s_1^{L*}, I)C_1(r_2^{L*}). \tag{13}$$

Substituting Equation (13) into Equation (11), I obtain Equation (12) again. If $r_1^L > s_1^{L*}$, substitute Equation (12) into Equation (13) to get

$$C_1(r_1^L) = W^L(s_1^{L*}, I)C_1(r_2^{L*}). \tag{14}$$

Since $W^L(s_1^{L*}, I) < 1$, this requires that $C_1(r_2^{L*}) > C_1(r_1^L)$, and thus, $r_2^{L*} > r_1^L$ (if $r_1^L > s_1^{L*}$).

Going through the same process to find r_1^H and its accompanying conditions, if $r_1^H > s_1^{H*}$, then

$$C_1(r_1^H) = W^H(s_1^{H*}, I)C_1(r_2^{H*})$$

and $r_2^{H*} > r_1^H$. Since by assumption, $r_L > r_H$ (the monies raised and spent if there were only one election and the incumbent knew the quality of his challenger was low or high, respectively), then the result (from Equation (7)) that $C_1(r_2^{L*}) > W_1^L(s_1^{L*}, I)$ yields $s_1^{L*} > r_2^{L*}$. But this means that $s_1^{L*} > r_2^{L*} > r_1^L$, and since $s_1^{L*} \leq r_1^L$, this yields a contradiction. Therefore, $r_1^L = s_1^{L*}$: an incumbent would never have a war chest if he knew that he was facing a high quality challenger in this election.

Now I show that $r_1^H < r_1^L$. There are two cases: 1-$\{r_1^H = s_1^{H*}\}$and 2-$\{r_1^H > s_1^{H*}\}$ (knowing that $r_1^L = s_1^{L*}$).

A.5.1 Case 1:

I wish to show that $r_1^H < r_1^L$ when $r_1^H = s_1^{H*}$. Suppose not. Then $r_1^H \geq r_1^L$, which implies that $C_1(r_1^H) \geq C_1(r_1^L)$. Substituting in the reduced first-order conditions from Equation (12),

$$W_1^H(r_1^H, I)\left[1 - C(r_2^*) + \eta_2 W^H(r_2^*, I) + (1-\eta_2)W^L(r_2^*, I)\right] \geq$$
$$W_1^L(r_1^L, I)\left[1 - C(r_2^*) + \eta_2 W^H(r_2^*, I) + (1-\eta_2)W^L(r_2^*, I)\right]$$

where $r_2^* = \bar{r}$ of Section A.2.2. Canceling out:

$$W_1^H(r_1^H, I) \geq W_1^L(r_1^L, I).$$

This yields a contradiction to the assumption that $r_L > r_H$.[4] Thus, $r_1^H < r_1^L$ (if there is no war chest).

A.5.2 Case 2:

I wish to show that $r_1^H < r_1^L$ when $r_1^H > s_1^{H*}$. Since $r_1^L = s_1^{L*}$, the incumbent brings no war chest with him into the second election cycle (if he draws a low quality challenger in the first election), and $r_2^{L*} = \bar{r}$. From Equation (12), $C_1(r_1^L) > W_1^L(r_1^L, I)$. This implies that $r_1^L > r_L > \bar{r}$. Since $r_1^H > s_1^{H*}$, $r_2^{H*} < \bar{r}$. From Equation (14), $r_2^{H*} > r_1^H$, and thus $r_1^H < \bar{r}$. Therefore $r_1^H < r_1^L$.

Since $\eta \in (0,1)$, by the concavity of the overall utility function, one obtains $r_1^H < r_1^* < r_1^L$. Now I examine when to expect a war chest. If the incumbent knew he was running against a low quality challenger, he would raise r_1^L (and spend $s_1^{L*} = r_1^L$). From the point of view of the model, he is in Case 3, and thus, $\mu_2 > 0$. Since $\frac{d\mu_2^*}{dr_1} < 0$ (from Equation (6)), as one decreases r_1 from r_1^L to r_1^*, μ_2^* increases, and thus, there is no war chest if an incumbent found out he was running against a low quality challenger *after* he raised the money. If the incumbent knew he was running against a high quality challenger, he would raise r_1^H. He may save some or none of it in this case. If he had no war chest in this case, since $\frac{d\mu_2^*}{dr_1} < 0$, as one increases r_1 from r_1^H to r_1^*, μ_2^* decreases—perhaps reaching 0, in which case there would be a war chest. If the incumbent had a war chest at r_1^H, then since $\frac{ds_1^{H*}}{dr_1^H} < 1$ (from Equation (9)), as one increased r_1 from r_1^H to r_1^*, the incumbent would keep a larger and larger war chest.□

[4]Or, more specifically, this yields a contradition to the shapes that $W^L(\cdot)$ and $W^H(\cdot)$ must take to maintain the assumption that $r_L > r_H$.

A.6 Relationship between incumbent strength and money

I utilize an application of the implicit function theorem for simultaneous equations (Chiang 1984, 210-212). The equations used are the first order conditions derived above: Equations 2 (twice: once for each challenger quality), 8 (twice: once for each challenger quality), and 10 (substituting in $r_1 = r_1^*$ to form the identity and setting it equal to 0). Dropping the * for simplicity:

$$\begin{array}{lll}
F^1: & -C_1(r_2^H) + \eta_2 W_1^H(r_1 - s_1^H + r_2^H, I) + (1-\eta_2) W_1^L(r_1 - s_1^H + r_2^H, I) & = 0 \\
F^2: & -C_1(r_2^L) + \eta_2 W_1^H(r_1 - s_1^L + r_2^L, I) + (1-\eta_2) W_1^L(r_1 - s_1^L + r_2^L, I) & = 0 \\
F^3: & W_1^H(s_1^H, I)\left\{ \begin{array}{r} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} - W^H(s_1^H, I)\left\{C_1(r_2^H)\right\} & = 0 \\
F^4: & W_1^L(s_1^L, I)\left\{ \begin{array}{r} 1 - C(r_2^L) + \eta_2 W^H(r_1 - s_1^L + r_2^L, I) \\ +(1-\eta_2) W^L(r_1 - s_1^L + r_2^L, I) \end{array} \right\} - W^L(s_1^L, I)\left\{C_1(r_2^L)\right\} & = 0 \\
F^5: & \left\{ \begin{array}{ll} -C_1(r_1) & +\eta_1 W_1^H(s_1^H, I)\left[\begin{array}{r} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right] \\ & +(1-\eta_1) W_1^L(s_1^L, I)\left[\begin{array}{r} 1 - C(r_2^L) + \eta_2 W^H(r_1 - s_1^L + r_2^L, I) \\ +(1-\eta_2) W^L(r_1 - s_1^L + r_2^L, I) \end{array} \right] \end{array} \right\} & = 0
\end{array}$$

Assuming continuous partial derivatives and a nonzero Jacobian determinant, through the implicit function theorem, we derive the following system:

$$\begin{bmatrix}
\frac{\partial F^1}{\partial r_1} & \frac{\partial F^1}{\partial s_1^H} & \frac{\partial F^1}{\partial s_1^L} & \frac{\partial F^1}{\partial r_2^H} & \frac{\partial F^1}{\partial r_2^L} \\
\frac{\partial F^2}{\partial r_1} & \frac{\partial F^2}{\partial s_1^H} & \frac{\partial F^2}{\partial s_1^L} & \frac{\partial F^2}{\partial r_2^H} & \frac{\partial F^2}{\partial r_2^L} \\
\frac{\partial F^3}{\partial r_1} & \frac{\partial F^3}{\partial s_1^H} & \frac{\partial F^3}{\partial s_1^L} & \frac{\partial F^3}{\partial r_2^H} & \frac{\partial F^3}{\partial r_2^L} \\
\frac{\partial F^4}{\partial r_1} & \frac{\partial F^4}{\partial s_1^H} & \frac{\partial F^4}{\partial s_1^L} & \frac{\partial F^4}{\partial r_2^H} & \frac{\partial F^4}{\partial r_2^L} \\
\frac{\partial F^5}{\partial r_1} & \frac{\partial F^5}{\partial s_1^H} & \frac{\partial F^5}{\partial s_1^L} & \frac{\partial F^5}{\partial r_2^H} & \frac{\partial F^5}{\partial r_2^L}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial r_1}{\partial I} \\ \frac{\partial s_1^H}{\partial I} \\ \frac{\partial s_1^L}{\partial I} \\ \frac{\partial r_2^H}{\partial I} \\ \frac{\partial r_2^L}{\partial I}
\end{bmatrix}
=
\begin{bmatrix}
-\frac{\partial F^1}{\partial I} \\ -\frac{\partial F^2}{\partial I} \\ -\frac{\partial F^3}{\partial I} \\ -\frac{\partial F^4}{\partial I} \\ -\frac{\partial F^5}{\partial I}
\end{bmatrix}.$$

The signs of the elements of the Jacobian are generally easily found:

$$\begin{bmatrix}
- & + & 0 & - & 0 \\
- & 0 & + & 0 & - \\
+ & - & 0 & - & 0 \\
+ & 0 & - & 0 & - \\
? & - & - & 0 & 0
\end{bmatrix}.$$

The one unknown, $\frac{\partial F^5}{\partial r_1}$, can be signed as negative given the assumption of $C_{11}(r) > [C_1(r)]^2$ (this is a sufficient, not necessary, condition). From this, the Jacobian determinant is negative. We can now find the sign of $\frac{\partial r_1}{\partial I}$ through Cramer's rule:

$$\frac{\partial r_1}{\partial I} = \frac{\begin{vmatrix}
-\frac{\partial F^1}{\partial I} & \frac{\partial F^1}{\partial s_1^H} & \frac{\partial F^1}{\partial s_1^L} & \frac{\partial F^1}{\partial r_2^H} & \frac{\partial F^1}{\partial r_2^L} \\
-\frac{\partial F^2}{\partial I} & \frac{\partial F^2}{\partial s_1^H} & \frac{\partial F^2}{\partial s_1^L} & \frac{\partial F^2}{\partial r_2^H} & \frac{\partial F^2}{\partial r_2^L} \\
-\frac{\partial F^3}{\partial I} & \frac{\partial F^3}{\partial s_1^H} & \frac{\partial F^3}{\partial s_1^L} & \frac{\partial F^3}{\partial r_2^H} & \frac{\partial F^3}{\partial r_2^L} \\
-\frac{\partial F^4}{\partial I} & \frac{\partial F^4}{\partial s_1^H} & \frac{\partial F^4}{\partial s_1^L} & \frac{\partial F^4}{\partial r_2^H} & \frac{\partial F^4}{\partial r_2^L} \\
-\frac{\partial F^5}{\partial I} & \frac{\partial F^5}{\partial s_1^H} & \frac{\partial F^5}{\partial s_1^L} & \frac{\partial F^5}{\partial r_2^H} & \frac{\partial F^5}{\partial r_2^L}
\end{vmatrix}}{\begin{vmatrix}
\frac{\partial F^1}{\partial r_1} & \frac{\partial F^1}{\partial s_1^H} & \frac{\partial F^1}{\partial s_1^L} & \frac{\partial F^1}{\partial r_2^H} & \frac{\partial F^1}{\partial r_2^L} \\
\frac{\partial F^2}{\partial r_1} & \frac{\partial F^2}{\partial s_1^H} & \frac{\partial F^2}{\partial s_1^L} & \frac{\partial F^2}{\partial r_2^H} & \frac{\partial F^2}{\partial r_2^L} \\
\frac{\partial F^3}{\partial r_1} & \frac{\partial F^3}{\partial s_1^H} & \frac{\partial F^3}{\partial s_1^L} & \frac{\partial F^3}{\partial r_2^H} & \frac{\partial F^3}{\partial r_2^L} \\
\frac{\partial F^4}{\partial r_1} & \frac{\partial F^4}{\partial s_1^H} & \frac{\partial F^4}{\partial s_1^L} & \frac{\partial F^4}{\partial r_2^H} & \frac{\partial F^4}{\partial r_2^L} \\
\frac{\partial F^5}{\partial r_1} & \frac{\partial F^5}{\partial s_1^H} & \frac{\partial F^5}{\partial s_1^L} & \frac{\partial F^5}{\partial r_2^H} & \frac{\partial F^5}{\partial r_2^L}
\end{vmatrix}}.$$

The determinant in the numerator has the following signs:

$$\begin{vmatrix} - & + & 0 & - & 0 \\ - & 0 & + & 0 & - \\ ? & - & 0 & - & 0 \\ ? & 0 & - & 0 & - \\ - & - & - & 0 & 0 \end{vmatrix}.$$

Through manipulation and substitution of the actual partial derivatives of the equations, it can be shown that the determinant is negative irrespective of the unknown signs. Thus, we find that $\frac{\partial r_1}{\partial I} > 0$. Through similar applications of Cramer's rule, it can be shown that all of the comparative statics in this system ($\frac{\partial s_1^H}{\partial I}, \frac{\partial r_2^L}{\partial I}$, etc.) are positive. A similar exercise can be conducted for when there are no war chests (in which case the number of equations reduces to three).□

A.7 Relationship between incumbent strength and war chests

In this model, the war chest is $r_1 - s_1^j$. Thus, we wish to find the sign of $\frac{\partial(r_1 - s_1^j)}{\partial I}$ or $\frac{\partial r_1}{\partial I} - \frac{\partial s_1^j}{\partial I}$. A sufficient, but not necessary, condition for this is that $\frac{\partial F^3}{\partial I}$ and $\frac{\partial F^4}{\partial I}$ both be negative. We find that

$$\begin{aligned} \frac{\partial F^3}{\partial I} = \; & W_{12}^H(s_1^H, I) \left\{ \begin{array}{r} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \\ & + W_1^H(s_1^H, I) \left\{ \begin{array}{r} \eta_2 W_2^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W_2^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \\ & - W_2^H(s_1^H, I) \left\{ C_1(r_2^H) \right\}. \end{aligned}$$

Since the first two terms are positive, and the third negative, we need to show that

$$\begin{aligned} W_2^H(s_1^H, I) \left\{ C_1(r_2^H) \right\} > \; & W_{12}^H(s_1^H, I) \left\{ \begin{array}{r} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \\ & + W_1^H(s_1^H, I) \left\{ \begin{array}{r} \eta_2 W_2^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W_2^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \end{aligned}$$

or

$$\begin{aligned} C_1(r_2^H) > \; & \frac{W_{12}^H(s_1^H, I)}{W_2^H(s_1^H, I)} \left\{ \begin{array}{r} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \\ & + W_1^H(s_1^H, I) \frac{\left\{ \begin{array}{r} \eta_2 W_2^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W_2^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\}}{W_2^H(s_1^H, I)} \end{aligned}$$

From the assumption $W_2(s_1, I) >> W_{12}(s_1, I)$, the first term on the right-hand side is close to zero. From the arguments above, $s_1^H > r_1 - s_1^H + r_2^H$, which implies that the fraction of the second line is less than one. Since $C_1(r_2^H) > W_1^H(s_1^H, I)$, the inequality holds. A similar argument can be made for $\frac{\partial F^4}{\partial I}$.□

B Proof of One Election Equilibrium

I show that ρ^* and α^* are mutual best responses.

B.1 Step 1: α^* is a best response to ρ^*

Incumbents in the lower region ($I < \hat{I}$) reveal their strength, and the challenger is better off running against them than staying out. For incumbents in the upper region ($I \geq \hat{I} \geq \iota$), the challenger updates her beliefs according to Bayes' Rule:

$$\mu^*(\breve{I}|\bar{r}) = \mu^*(\tilde{I}|\bar{r}) > 0, \forall \breve{I}, \tilde{I} \in [\hat{I}, \bar{I}].$$

The (updated) expected payoff from the first-election pooling incumbents is therefore

$$E[1 - W^H(s, I)|\bar{r}] = 1 - \int_{\hat{I}}^{\bar{I}} g(I)W^H(\bar{r}, I)dI \leq c^H.$$

For the case that $c^H = 1 - \int_{\hat{I}}^{\bar{I}} g(I)W^H(\bar{r}, I)dI$, the challenger is indifferent between entering and not entering, and thus has no incentive to deviate from the equilibrium of not entering. For the case that $c^H > 1 - \int_{\hat{I}}^{\bar{I}} g(I)W^H(\bar{r}, I)dI$, the challenger strictly prefers staying out to entering.

B.2 Step 2: ρ^* is a best response to α^*

The incumbents in the upper region ($I \geq \hat{I}$) have no incentive to spend, raise, and spend *more* money than $\bar{r}$ in the election—they may still deter the high quality challenger from entering, but they decrease their utility. Thus, they have no incentive to deviate. If the same incumbents spend and raise *less* money, then the high quality challenger enters—the utility gain they obtain by spending and raising less is more than offset by the utility loss from the high quality challenger entering. Thus, they have no incentive to deviate. Incumbents in the lower region ($I < \hat{I}$) are all maximizing utility under the assumption that a high quality challenger will enter in the second election. The only way to deter that challenger is by imitating the incumbents in the upper region. But the cost of imitating is greater than the benefit of deterring the high quality challenger. Thus, they do not deviate.

B.3 Universal Divinity

To see that this equilibrium satisfies Universal Divinity, I examine the "most likely" incumbent to choose any out-of-equilibrium action. Consider r. I examine all r that occur out of equilibrium. If the challenger were to observe $\tilde{r} < r(\underline{I})$, the challenger's belief would be $\mu^*(\underline{I}|\tilde{r}) = 1$, that is, the challenger would believe that the weakest incumbent would take such an action. If it would be beneficial for any $I > \underline{I}$ to raise $\tilde{r}_1$, it would also be beneficial for $\underline{I}$. Thus, $\underline{I}$ is the "most likely" incumbent to take this action. Let $r(\hat{I})$ be the r raised by incumbent $I = \hat{I}$ if that incumbent knew that high quality challengers would run against him in the second election. If the challenger were to observe $\dot{r} \in [r(\hat{I}), \bar{r})$, the universally divine belief would be $\mu^*(I = \hat{I}|\dot{r}) = 1$. In fact, the incumbent $I = \hat{I}$ is indifferent between raising $r(I = \hat{I})$ (and running against high quality challengers in the second election) and raising $\bar{r}$ (and running against a low quality challenger in the second election). If it would be beneficial for any $I \neq \hat{I}$ to raise $\dot{r}$, it would also be beneficial for $I = \hat{I}$. Thus, $I = \hat{I}$ is the "most likely" to take this action. If the challenger were to observe $\breve{r} > \bar{r}$, the challenger's belief would be $\mu^*(\overline{I}|\breve{r}) = 1$, that is, the challenger would believe that the strongest incumbent would take such an action. If it would be beneficial for any $I < \overline{I}$ to raise $\breve{r}$, it would also be beneficial for $\overline{I}$. Thus, $\overline{I}$ is the "most likely" incumbent to take this action.

B.4 No other equilibria

To see that there are no universally divine pooling equilibria, suppose not. In a pooling equilibrium, all incumbents raise the same $r = \ddot{r}$.Since the challenger cannot distinguish between incumbents, her beliefs are

$$\mu^*(\breve{I}|\ddot{r}) = \mu^*(\tilde{I}|\ddot{r}) > 0, \forall \breve{I}, \tilde{I} \in [\underline{I}, \overline{I}],$$

and the expected payoff from the pooling incumbents is

$$E[1 - W^H(s, I)|\ddot{r}] = 1 - \int_{\underline{I}}^{\bar{I}} g(I)W^H(\ddot{r}, I)dI < c^H.$$ [5]

The challenger's best response is to enter the election. If this were an equilibrium, Universal Divinity requires that for any $\breve{r}_1 > \ddot{r}$, $\mu^*(\overline{I}|\breve{r}_1) = 1$. But since the challenger will never run against $\overline{I}$, incumbent $\overline{I}$ would do better to choose some $r_1 = \ddot{r} + \varepsilon$, and thus destroy the equilibrium.

[5] If $1 - \int_{\underline{I}}^{\bar{I}} g(I)W^H(\ddot{r}, I)dI \geq c^H$, then some incumbents would rather not pool, but separate and reveal their types.

To see that there are no separating equilibria, suppose not. In a separating equilibrium, all incumbents raise different r and reveal their strength. Challengers run against all I such that $c^H < 1 - W^H(r, I)$ and not against $c^H > 1 - W^H(r, I)$. Incumbents raise (and spend) money according to values found in the two-election, perfect-information equilibrium. But incumbent $I = \hat{I} - \varepsilon$, who runs against the high quality challenger, would be better off imitating incumbent $I = \hat{I} + \varepsilon$, who runs against the low quality challenger. Thus, this is not an equilibrium.

There are several other semi-pooling possibilities that are not universally divine equilibria. Suppose that there is a semi-pooling equilibrium as described in the text, but with $I < \iota$. Then the challenger would knowingly run against incumbent strengths that are stronger than her, and this is not an equilibrium. Suppose that there is a semi-pooling equilibrium as described above, but with $\hat{I}$ such that $c^H > 1-W^H(r, I)$. Then when the upper region incumbents raise the same money $\dot{r}$, the challenger's updated payoff from those incumbents is

$$E[1 - W^H(s, I)|\dot{r}] = 1 - \int_{\hat{I}}^{\bar{I}} g(I)W^H(\dot{r}, I)dI > c^H,$$

and the challenger enters the race. Following the same logic as why there are no pooling equilibria, the strongest incumbent would want to raise $r_1 = \dot{r}_1 + \varepsilon$, and this would destroy the equilibrium.

There are never any lower regions that pool, since a high quality challenger would run against those incumbents, and if that were the case, those incumbents would be better off choosing $\check{r}(I)$ as described in the proposition. There are never two separate pooling regions—one of the regions would either be fully above ι or below ι. And in either case, the incumbents in that region would be better off separating. If the region is below ι (and thus, the challenger would enter against them), the incumbents separate for the same reason that there are no lower pooling regions. If the region is above ι (and thus, the challenger would not enter against them), those incumbents would be better off revealing their strength (the challenger would still not enter since the universally divine out-of-equilibrium challenger belief would be that the action was taken by an incumbent in that upper region). There are no middle regions that pool (with separating regions on either side) as it is not possible to make the incumbents at either side of the pooling region indifferent—if an incumbent at the edge of the pooling region strictly prefers to pool, then his adjacent incumbent who is separating would also rather pool, thus destroying the equilibrium.□

B.5 Characterizing the Equilibrium

In this model, there are three values that parameterize the equilibrium, which are found by solving three nonlinear equations. First, for any weak incumbent (who expects high quality challenger entry), the marginal cost of fund-raising must equal the marginal benefit. This is particularly true for the cutpoint incumbent, $\hat{I}$:

$$C_1(r(\hat{I})) = W_1^H(r(\hat{I}), \hat{I}).$$

Second, the cutpoint incumbent must be indifferent between separating with weak incumbents (and drawing the high quality challenger) and pooling with strong incumbents (and drawing the low quality challenger):

$$-C(r(\hat{I})) + W^H(r(\hat{I}), \hat{I}) = -C(\bar{r}) + W^L(\bar{r}, \hat{I}).$$

Finally, the challenger must weakly prefer not to enter against the (strong) pooling incumbents:

$$c^H \geq 1 - \int_{\hat{I}}^{\bar{I}} g(I)W^H(\bar{r}, I)dI.$$

When the third equation is set to equality, with three equations and three unknowns, one can solve for $\hat{I}$, $r(\hat{I})$, and $\bar{r}$. Any $r > \bar{r}$ can then be chosen, and $\hat{I}$, $r(\hat{I})$ solved for.

C Proof of Two Election with No Uncertainty Equilibrium

Definition. A universally divine sequential equilibrium to the game is a set of strategy pairs (ρ^*, σ^*), (α^*, β^*) and beliefs (μ^*, ν^*) where

(a) $\rho^*(I)$ maximizes $EU_i(r_1, I|\sigma^*, \alpha^*, \beta^*), \forall I \in [\underline{I}, \overline{I}]$,
(b) $\alpha^*(r_1)$ maximizes
$EU_c(\{\text{enter, not enter}\}, r_1|\rho^*, \sigma^*, \beta^*, \mu^*), \forall r_1 \in \mathbb{R}_+$,
(c) $\sigma^*(\{\text{enter, not enter}\}, I)$ maximizes
$EU_i(s_1^j, r_2^j, \{\text{enter, not enter}\}, I|\rho^*, \alpha^*, \beta^*), \forall I \in [\underline{I}, \overline{I}]$,
(d) $\beta^*(r_1, s_1^j, r_2^j; j)$ maximizes
$EU_c(\{\text{enter, not enter}\}, r_1, s_1^j, r_2^j; j|\rho^*, \sigma^*, \alpha^*, \nu^*), \forall (r_1, s_1^j, r_2^j; j) \in \mathbb{R}_+^3 \times \{H, L\}$,
(e) if $\rho^{*-1}(r_1) \neq \varnothing$, then $\mu^*(I|r_1)$ is derived via
Bayes' Rule as defined by ρ^* and $f(I)$,
(f) if $\rho^{*-1}(r_1) = \varnothing$, then $\mu^*(I|r_1) > 0$ only for I that
are "most likely" to deviate from the equilibrium
(g) if $\sigma^{*-1}(r_1, s_1^j, r_2^j) \neq \varnothing$, then $\nu^*(I|r_1, s_1^j, r_2^j; j)$
is derived via Bayes' Rule as defined by ρ^*, σ^* and $f(I)$, and
(h) if $\sigma^{*-1}(r_1, s_1^j, r_2^j) = \varnothing$, then $\nu^*(I|r_1, s_1, r_2; j) > 0$
only for I that are "most likely" to deviate from the equilibrium.

C.1 Separating Equilibrium

For a separating equilibrium, incumbents must have no incentive to imitate each other. This is straightforward in the lower region where incumbents maximize expecting high quality challengers to enter in both elections: the comparative statics presented above hold. This is less straightforward in the upper region where incumbents not only maximize expecting low quality challenger to enter in both elections, but must make such challengers indifferent to entering *and* the cut-point incumbent indifferent to imitating. Let $\dot{I} < \ddot{I}$. In equilibrium, $\dot{r}_1 < \ddot{r}_1$, $\dot{s}_1^L < \ddot{s}_1^L$, $\dot{r}_2^L > \ddot{r}_2^L$, and $\dot{s}_2^L > \ddot{s}_2^L$. The incentive compatibility conditions are, for all $I > \hat{I}$:

$$\begin{aligned}
-C(\ddot{r}_1) + W^L(\ddot{s}_1^L, \ddot{I}) \left[1 - C(\ddot{r}_2^L) + W^L(\ddot{s}_2^L, \ddot{I})\right] &\geq -C(\dot{r}_1) + W^L(\dot{s}_1^L, \ddot{I}) \left[1 - C(\dot{r}_2^L) + W^L(\dot{s}_2^L, \ddot{I})\right] \\
-C(\dot{r}_1) + W^L(\dot{s}_1^L, \dot{I}) \left[1 - C(\dot{r}_2^L) + W^L(\dot{s}_2^L, \dot{I})\right] &\geq -C(\ddot{r}_1) + W^L(\ddot{s}_1^L, \dot{I}) \left[1 - C(\ddot{r}_2^L) + W^L(\ddot{s}_2^L, \dot{I})\right].
\end{aligned}$$

Subtracting the RHS of the second inequality from the LHS of the first inequality, and subtracting LHS of the second inequality from the RHS of the first inequality:

$$\begin{aligned}
&W^L(\ddot{s}_1^L, \ddot{I}) \left[1 - C(\ddot{r}_2^L) + W^L(\ddot{s}_2^L, \ddot{I})\right] - \left\{W^L(\ddot{s}_1^L, \dot{I}) \left[1 - C(\ddot{r}_2^L) + W^L(\ddot{s}_2^L, \dot{I})\right]\right\} \\
&\geq W^L(\dot{s}_1^L, \ddot{I}) \left[1 - C(\dot{r}_2^L) + W^L(\dot{s}_2^L, \ddot{I})\right] - \left\{W^L(\dot{s}_1^L, \dot{I}) \left[1 - C(\dot{r}_2^L) + W^L(\dot{s}_2^L, \dot{I})\right]\right\}.
\end{aligned}$$

Rearranging, we get

$$\begin{aligned}
&\left[W^L(\ddot{s}_1^L, \ddot{I}) - W^L(\ddot{s}_1^L, \dot{I})\right] \left[1 - C(\ddot{r}_2^L)\right] + W^L(\ddot{s}_1^L, \ddot{I}) W^L(\ddot{s}_2^L, \ddot{I}) - W^L(\ddot{s}_1^L, \dot{I}) W^L(\ddot{s}_2^L, \dot{I}) \\
&\geq \left[W^L(\dot{s}_1^L, \ddot{I}) - W^L(\dot{s}_1^L, \dot{I})\right] \left[1 - C(\dot{r}_2^L)\right] + W^L(\dot{s}_1^L, \ddot{I}) W^L(\dot{s}_2^L, \ddot{I}) - W^L(\dot{s}_1^L, \dot{I}) W^L(\dot{s}_2^L, \dot{I}).
\end{aligned}$$

Comparing the first bracketed expression on each side, by $W_{12}^L(s, I) > 0$ (the supermodularity condition),

$$\left[W^L(\ddot{s}_1^L, \ddot{I}) - W^L(\ddot{s}_1^L, \dot{I})\right] > \left[W^L(\dot{s}_1^L, \ddot{I}) - W^L(\dot{s}_1^L, \dot{I})\right].$$

By $C_1(r) > 0$,

$$\left[1 - C(\ddot{r}_2^L)\right] > \left[1 - C(\dot{r}_2^L)\right].$$

By Corollary 2.6.3 of Topkis (1998), the product of two nonnegative, increasing supermodular functions is itself supermodular. Since $W^L(s,I)$ is nonnegative and increasing, the function $W^L(s_1,I)W^L(s_2,I)$ is supermodular,which means that

$$W^L(\ddot{s}_1^L,\ddot{I})W^L(\ddot{s}_2^L,\ddot{I}) - W^L(\ddot{s}_1^L,\dot{I})W^L(\ddot{s}_2^L,\dot{I}) > W^L(\dot{s}_1^L,\ddot{I})W^L(\dot{s}_2^L,\ddot{I}) - W^L(\dot{s}_1^L,\dot{I})W^L(\dot{s}_2^L,\dot{I}).$$

Since each component of the LHS is greater than its equivalent component on the RHS (and is nonnegative), the incentive compatibility inequality holds.

C.2 Universal Divinity

To see that this equilibrium satisfies Universal Divinity, I examine the "most likely" incumbent to choose any out-of-equilibrium action. First, consider r_1. I examine all r_1 that occur out of equilibrium. If the challenger were to observe $\tilde{r}_1 < \check{r}_1(\underline{I})$, the challenger's belief would be $\mu^*(\underline{I}|\tilde{r}_1) = 1$, that is, the challenger would believe that the weakest incumbent would take such an action. If it would be beneficial for any $I > \underline{I}$ to raise $\tilde{r}_1$, it would also be beneficial for $\underline{I}$. Thus, $\underline{I}$ is the "most likely" incumbent to take this action. Recall that $\check{r}_1(\hat{I})$ is the r_1 raised by incumbent $I = \hat{I}$ anticipating that high quality challengers will run against him in both elections. Further, recall that $\hat{r}_1(\hat{I})$ is the amount raised by incumbent $I = \hat{I}$ to deter the high quality challenger and anticipating that low quality challengers will run against him in both elections. If the challenger were to observe $\dot{r}_1 \in (\check{r}_1(\hat{I}), \hat{r}_1(\hat{I})]$, the universally divine belief would be $\mu^*(I = \hat{I}|\dot{r}_1) = 1$. In fact, the incumbent $I = \hat{I}$ is indifferent between raising $\check{r}_1(\hat{I})$ (and running against high quality challengers in both elections) and raising $\hat{r}_1(\hat{I})$ (and running against low quality challengers in both elections). If it would be beneficial for any $I \neq \hat{I}$ to raise $\dot{r}_1$, it would also be beneficial for $I = \hat{I}$. Thus, $I = \hat{I}$ is the "most likely" to take this action. If the challenger were to observe $\breve{r}_1 > \hat{r}_1(\overline{I})$, the challenger's belief would be $\mu^*(\overline{I}|\breve{r}_1) = 1$, that is, the challenger would believe that the strongest incumbent would take such an action. If it would be beneficial for any $I < \overline{I}$ to raise $\breve{r}_1$, it would also be beneficial for $\overline{I}$. Thus, $\overline{I}$ is the "most likely" incumbent to take this action.

Now consider s_1. If the challenger were to observe $\tilde{s}_1^j < \check{s}_1^j(\underline{I})$, the challenger's belief would be $\nu^*(\underline{I}|r_1, \tilde{s}_1^j, r_2^j; j) = 1$, that is, the challenger would believe that the weakest incumbent would take such an action for the same reasons as $\tilde{r}_1$ above. Let $\check{s}_1^H(\hat{I})$ be the s_1 raised by incumbent $I = \hat{I}$ if that incumbent knew that high quality challengers would run against him in both elections. Also let $\hat{s}_1^L(\hat{I})$ be the $s_1$raised by incumbent $I = \hat{I}$ if that incumbent knew that low quality challengers would run against him in both elections. If the challenger were to observe $\dot{s}_1^j \in (\check{s}_1^H(\hat{I}), \hat{s}_1^L(\hat{I})]$, the universal divine belief would be $\nu^*(I = \hat{I}|r_1, \dot{s}_1^j, r_2^j; j) = 1$, for the same reasons as $\dot{r}_1$ above. If the challenger were to observe $\breve{s}_1^j > \hat{s}_1^L(\overline{I})$, the challenger's belief would be $\nu^*(\overline{I}|r_1, \breve{s}_1^j, r_2^j; j) = 1$, for the same reasons as $\breve{r}_1$ above. Similar universal divine beliefs are required for ν^* for out-of-equilibrium r_2.

C.3 No other equilibria

To see that there are no universally divine pooling equilibria, suppose not. In a pooling equilibrium, all incumbents raise the same $r_1 = \ddot{r}_1$. Since the challenger cannot distinguish between incumbents, her beliefs are

$$\mu^*(\breve{I}|\ddot{r}_1) = \mu^*(\tilde{I}|\ddot{r}_1) > 0, \forall \breve{I}, \tilde{I} \in [\underline{I}, \overline{I}],$$

and the expected payoff from the pooling incumbents is

$$E[1 - W^H(s,I)|\ddot{r}] = 1 - \int_{\underline{I}}^{\bar{I}} g(I)W^H(\ddot{r},I)dI < c^H.$$ [6]

The challenger's best response is to enter the election. If this were an equilibrium, Universal Divinity requires that for any $\breve{r}_1 > \ddot{r}$, $\mu^*(\overline{I}|\breve{r}_1) = 1$. But since the challenger will never run against $\overline{I}$, incumbent $\overline{I}$ would do better to choose some $r_1 = \ddot{r} + \varepsilon$, and thus destroy the equilibrium. If there is no pooling in the first election, there can be no pooling in the second election.

[6] If $1 - \int_{\underline{I}}^{\bar{I}} g(I)W^H(\ddot{r},I)dI \geq c^H$, then some incumbents would rather not pool, but separate and reveal their types.

There is no universally divine semi-pooling equilibrium like the one-election model. As discussed in the text, this is because of the characteristics of spending. Suppose there were an upper region of semi-pooling, which requires that stronger incumbents all raise, spend, and raise the same amount of money. Having all raised $\bar{r}_1$, the semi-pooling would require that all the same incumbents also spend $\bar{s}_1^L$. But the strongest incumbent wants to spend more of the money raised. Observing such an action, the challenger believes (by Universal Divinity) that the deviator is the strongest incumbent (correctly), and this breaks the equilibrium.

There are several other semi-pooling possibilities that are not universally divine equilibria. There are never any lower regions that pool, since a high quality challenger would run against those incumbents, and if that were the case, those incumbents would be better off choosing $\check{r}(I)$ as described in the proposition. There are never two separate pooling regions—one of the regions would either be fully above $\hat{I}$ or below $\hat{I}$. And in either case, the incumbents in that region would be better off separating. If the region is below $\hat{I}$ (and thus, the challenger would enter against them), the incumbents separate for the same reason that there are no lower pooling regions. If the region is above $\hat{I}$ (and thus, the challenger would not enter against them), some of those incumbents would be better off revealing their strength through their spending (the challenger would still not enter since the universally divine out-of-equilibrium challenger belief would be that the action was taken by an incumbent in that upper region). There are no middle regions that pool (with separating regions on either side) as it is not possible to make the incumbents at either side of the pooling region indifferent—if an incumbent at the edge of the pooling region strictly prefers to pool, then his adjacent incumbent who is separating would also rather pool, thus destroying the equilibrium.

There are no universally divine equilibria in which a set of incumbents pool (and hide their strength) in the first election, and then reveal their strength in the second election. Suppose not. First, all incumbents cannot initially pool on some $\bar{r}_1$ for reasons stated above. And no lower or middle region can initially pool, also for reasons given above. That leaves the possibility that (some or) all incumbents above $\hat{I}$ pool on some $\bar{r}_1$, and then separate on s_1. In any such equilibrium, the high quality challenger must still be deterred in the second election and no weaker incumbents (below $\hat{I}$) must want to imitate. The final constraint is to maximize utility subject to the other two constraints. However, by taking away one degree of freedom (r_1), it is no longer possible to satisfy all three requirements with the other two parameters (s_1, r_2).

There are no other universally divine separating equilibria. If the cutpoint incumbent ($\hat{I}$) is not indifferent to imitating strong incumbents, then he prefers to imitate—breaking the equilibrium—or strong incumbents are exerting too much effort (e.g. raising more money than is required. If the high quality challenger is not indifferent to entering, then, then either she will enter—breaking the equilibrium—or strong incumbents are again exerting too much effort. Thus, in any separating equilibrium, these two indifference conditions must hold, and the strong incumbents maximize their utility subject to these two restrictions. This is possible because there are three parameters that the incumbent chooses.□

D Proof of Two Election with Uncertainty Equilibrium

This follows the logic of the previous two proofs, and is omitted.

E References

Chiang, Alpha C. 1984. *Fundamental Methods of Mathematical Economics*, 3rd edition. New York: McGraw-Hill/Irwin.

Sundaram, Rangarajan K. 1996. *A First Course in Optimization Theory*. New York: Cambridge University Press.

Topkis, Donald M. 1998. *Supermodularity and Complementarity*. Princeton: Princeton University Press.