Appendix to "When Do War Chests Deter?"

Jay Goodliffe

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In this appendix, I sign the comparative statics—such as whether fundraising increases in incumbent strength—for the incumbent holding challenger quality constant, i.e. when challenger entry is not strategic. I then give a detailed proof for the perfect-information, no-uncertainty equilibrium presented in the main text.

A Proof of Comparative Statics

In this section, I present the solution to the incumbent's maximization problem holding (probability of) challenger quality constant. The method is to work backwards through each decision point of the incumbent solving each maximization problem with the Kuhn-Tucker method. I then show how war chests and other variables change with incumbent strength through the implicit funciton theorem.

Because $W_1(s, I) > 0$,[1] the incumbent will spend all that he has in the last election: $s_2 = r_1 - s_1 + r_2$ (recall that subscripts denote the first or second election). Since the probabilities of drawing a high quality challenger are held constant here (that is, they are known by the incumbent), the incumbent's utility maximization problem is a nested maximization problem:

$$\begin{aligned}
&\max_{r_1 \geq 0} -C(r_1) \\
&+\eta_1 \left\{ \max_{0 \leq s_1^H \leq r_1} W^H(s_1^H, I) \left\langle 1 + \max_{r_2^H \geq 0} \left[\begin{array}{c} -C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right] \right\rangle \right\} \\
&+(1-\eta_1) \left\{ \max_{0 \leq s_1^L \leq r_1} W^L(s_1^L, I) \left\langle 1 + \max_{r_2^L \geq 0} \left[\begin{array}{c} -C(r_2^L) + \eta_2 W^H(r_1 - s_1^L + r_2^L, I) \\ +(1-\eta_2) W^L(r_1 - s_1^L + r_2^L, I) \end{array} \right] \right\rangle \right\}
\end{aligned}$$

I solve the inner maximization problems and work outwards (to the overall maximization problem).

A.1 Solving for money raised in the second election

First solve:

$$\max_{r_2^j \geq 0} \left[-C(r_2^j) + \eta_2 W^H(r_1 - s_1^j + r_2^j, I) + (1-\eta_2) W^L(r_1 - s_1^j + r_2^j, I) \right]$$

(The method is the same whether a low or high quality challenger ran in the first election, *i.e.*, $j = L$ or H.) For this problem, $r_1 - s_1^j$ (the war chest) is a constant. Using the method of Kuhn-Tucker to solve this problem, I first set up the Lagrangian:

$$L(r_2^j, \lambda) = -C(r_2^j) + \eta_2 W^H(r_1 - s_1^j + r_2^j, I) + (1-\eta_2) W^L(r_1 - s_1^j + r_2^j, I) + \lambda r_2^j.$$

[1] Following convention, a subscript n on a function denotes the partial derivative of that function with respect to the nth argument of the function. Thus, $W_1(s, q) = \dfrac{\partial W(s, q)}{\partial s}$.

The first-order conditions are

$$\frac{\partial L}{\partial r_2^j} = -C_1(r_2^j) + \eta_2 W_1^H(r_1 - s_1^j + r_2^j, I) + (1-\eta_2)W_1^L(r_1 - s_1^j + r_2^j, I) + \lambda = 0$$
$$\lambda \geq 0, \quad r_2^j \geq 0, \quad \lambda r_2^j = 0.$$

There are two cases: 1-$\{\lambda > 0,\ r_2^j = 0\}$and 2-$\{\lambda = 0,\ r_2^j > 0\}$. I examine the cases separately.

A.1.1 Case 1:

Here $r_2^j = 0$. Therefore,

$$\eta_2 W_1^H(r_1 - s_1^j, I) + (1-\eta_2)W_1^L(r_1 - s_1^j, I) < C_1(0).$$

Since $C_1(0) = 0$ and $W_1^j(\cdot,\cdot) > 0$ by assumption, this cannot be true.

A.1.2 Case 2:

Here $\lambda = 0$. For a given war chest $(r_1 - s_1^j)$, find the $r_2^j > 0$ that solves

$$\eta_2 W_1^H(r_1 - s_1^j + r_2^j, I) + (1-\eta_2)W_1^L(r_1 - s_1^j + r_2^j, I) = C_1(r_2^j). \tag{1}$$

For future use I derive some comparative statics. Substituting the r_2^{j*} that solves Equation (1), I obtain

$$\eta_2 W_1^H(r_1 - s_1^j + r_2^{j*}, I) + (1-\eta_2)W_1^L(r_1 - s_1^j + r_2^{j*}, I) \equiv C_1(r_2^{j*}). \tag{2}$$

Partially differentiating both sides with respect to s_1^j, I obtain

$$\left\{\eta_2 W_{11}^H(r_1 - s_1^j + r_2^{j*}, I) + (1-\eta_2)W_{11}^L(r_1 - s_1^j + r_2^{j*}, I)\right\}\left\{-1 + \frac{\partial r_2^{j*}}{\partial s_1^j}\right\} = C_{11}(r_2^{j*})\frac{\partial r_2^{j*}}{\partial s_1^j}.$$

Since $W_{11}^j(\cdot,\cdot) < 0$ and $C_{11}(\cdot) > 0$, I find that $0 < \dfrac{\partial r_2^{j*}}{\partial s_1^j} < 1$. Looking further ahead, I substitute the r_2^{j*} that solves Equation (1) with the equilibrium s_1^{j*}:

$$\eta_2 W_1^H(r_1 - s_1^{j*} + r_2^{j*}, I) + (1-\eta_2)W_1^L(r_1 - s_1^{j*} + r_2^{j*}, I) \equiv C_1(r_2^{j*}). \tag{3}$$

Differentiating both sides with respect to r_1, I obtain

$$\left\{\begin{array}{c} \eta_2 W_{11}^H(r_1 - s_1^{j*} + r_2^{j*}, I) + \\ (1-\eta_2)W_{11}^L(r_1 - s_1^{j*} + r_2^{j*}, I) \end{array}\right\}\left\{1 - \frac{ds_1^{j*}}{dr_1} + \frac{dr_2^{j*}}{dr_1}\right\} = C_{11}(r_2^{j*})\frac{dr_2^{j*}}{dr_1}.$$

Since $W_{11}(\cdot,\cdot) < 0$ and $C_{11}(\cdot) > 0$, I find

$$\begin{array}{lll} \dfrac{dr_2^{j*}}{dr_1} > 0 & \Leftrightarrow & \dfrac{ds_1^{j*}}{dr_1} > 1 + \dfrac{dr_2^{j*}}{dr_1} > 1 \\ \dfrac{dr_2^{j*}}{dr_1} = 0 & \Leftrightarrow & \dfrac{ds_1^{j*}}{dr_1} = 1 \\ \dfrac{dr_2^{j*}}{dr_1} < 0 & \Leftrightarrow & \dfrac{ds_1^{j*}}{dr_1} < 1 + \dfrac{dr_2^{j*}}{dr_1} < 1. \end{array} \tag{4}$$

A.2 Solving for the money spent in the first election

Now, solve the next maximization problem—finding s_1^j:

$$\max_{0\leq s_1^j\leq r_1} W^j(s_1^j, I)\left[1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) + (1-\eta_2)W^L(r_1 - s_1^j + r_2^{j*}, I)\right].$$

(The maximization problem is similar for s_1^L and s_1^H). In this problem, r_1 is treated as a constant. Solve again through the method of Kuhn-Tucker by first setting up the Lagrangian:

$$\begin{aligned} L(s_1^j, \mu_1, \mu_2) = \;& W^H(s_1^j, I)\left\{ \begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^j + r_2^{j*}, I) \end{array} \right\} \\ & +\mu_1 s_1^j + \mu_2(r_1 - s_1^j). \end{aligned}$$

The first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial s_1^j} = \;& W_1^j(s_1^j, I)\left\{ \begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^j + r_2^{j*}, I) \end{array} \right\} \\ & +W^j(s_1^j, I)\left\{ \begin{array}{l} -C_1(r_2^{j*})\dfrac{\partial r_2^{j*}}{\partial s_1^j} \\ + \left[\begin{array}{l} \eta_2 W_1^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W_1^L(r_1 - s_1^j + r_2^{j*}, I) \end{array} \right] \left[-1 + \dfrac{\partial r_2^{j*}}{\partial s_1^j}\right] \end{array} \right\} \\ & +\mu_1 - \mu_2 \\ = \;& 0 \end{aligned}$$

$$\mu_1 s_1^j = 0, \quad \mu_1 \geq 0, \quad s_1^j \geq 0$$
$$\mu_2(r_1 - s_1^j) = 0, \quad \mu_2 \geq 0, \quad r_1 \geq s_1^j.$$

Using Equation (2), substitute and simplify, and obtain

$$\begin{aligned} W_1^j(s_1^j, I)&\left\{ \begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^j + r_2^{j*}, I) \end{array} \right\} \\ &+W^j(s_1^j, I)\left\{-C_1(r_2^{j*})\right\} \\ &+\mu_1 - \mu_2 = 0. \end{aligned}$$

There are four cases: 1-$\{\mu_1 > 0, \mu_2 > 0\}$, 2-$\{\mu_1 > 0, \mu_2 = 0\}$, 3-$\{\mu_1 = 0, \mu_2 > 0\}$, 4-$\{\mu_1 = 0, \mu_2 = 0\}$. I examine each of these in turn.

A.2.1 Case 1:

Since $\mu_1 > 0$ and $\mu_2 > 0$, $r_1 = s_1^j = 0$. This can only be true if $r_1 = 0$, in which case, this is the only possible solution. If $r_1 > 0$, this is not possible.

A.2.2 Case 2:

Here $\mu_1 > 0$ and $\mu_2 = 0$. Thus, $s_1^j = 0$. For this to be true,

$$W_1^j(0, I)\left\{ \begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 + r_2^{j*}, I) \end{array} \right\} + \mu_1 = W^j(0, I)\left\{C_1(r_2^{j*})\right\}.$$

Since $\mu_1 > 0$,

$$W_1^j(0,I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 + r_2^{j*}, I) \end{array}\right\} < W^j(0,I)\left\{C_1(r_2^{j*})\right\}.$$

Rearranging,

$$\frac{W_1^j(0,I)}{C_1(r_2^{j*})} < \frac{W^j(0,I)}{\left\{1 - C(r_2^{j*}) + \eta_2 W^H(r_1 + r_2^{j*}, I) + (1-\eta_2)W^L(r_1 + r_2^{j*}, I)\right\}}. \tag{5}$$

To see that this is not possible, first define $\bar{r}$ as the r that solves $C_1(r) = pW_1^H(r,I) + (1-p)W_1^L(r,I)$.[2] This would be the r_2 chosen by the incumbent if no war chest were brought from the first election ($r_1 = s_1$). This $\bar{r}$ exists by the assumptions that $W_1^j(0,I) > C_1(0)$ and there exists some $\tilde{r}$ such that $W^j(\tilde{r},I) > C(\tilde{r})$—$\bar{r}$ will be one of those $\tilde{r}$. From the first maximization problem solved (Equation (2)), $\eta_2 W_1^H(r_1 + r_2^{j*}, I) + (1-\eta_2)W_1^L(r_1 + r_2^{j*}, I) = C_1(\breve{r}_2^{j*})$, where $\breve{r}_2^{j*}$ is the specific r_2^{j*} that solves this equation. From the restrictions on the first and second derivatives of $W^j(\cdot,\cdot)$ and $C(\cdot)$, I find that $\breve{r}_2^{j*} < \bar{r} < r_1 + \breve{r}_2^{j*}$ (remembering that r_1 must be positive, else this case is not possible). Since there exists some $\tilde{r}$ such that $W^j(\tilde{r},I) > C(\tilde{r})$, there also exists an $\bar{r}$ such that $pW^H(\bar{r},I) + (1-p)W^L(\bar{r},I) > C(\bar{r})$. This implies that $\eta_2 W^H(r_1 + r_2^{j*}, I) + (1-\eta_2)W^L(r_1 + r_2^{j*}, I) > C(r_2^{j*})$. Hence the denominator on the RHS of Equation (5) is greater than 1, while the numerator on the RHS is less than 1. Thus the fraction on the RHS is less than 1. From the Inada-type conditions on $W_1^j(\cdot,\cdot)$, note that the LHS is greater than 1. Thus, the inequality cannot hold, and this case is not possible.

A.2.3 Case 3:

In this case, $\mu_1 = 0$ and $\mu_2 > 0$, which implies that $s_1^j = r_1 > 0$. This yields the first order condition:

$$W_1^j(r_1,I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_2^{j*}, I) \end{array}\right\} = W^j(r_1,I)\left\{C_1(r_2^{j*})\right\} + \mu_2.$$

Assuming that the solution is not at a corner, this can be made into an identity:

$$W_1^j(r_1,I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_2^{j*}, I) \end{array}\right\} \equiv W^j(r_1,I)\left\{C_1(r_2^{j*})\right\} + \mu_2^*.$$

Since $s_1^j = r_1$, $\frac{ds_1^{j*}}{dr_1} = 1$, which implies (from Equation (4)) $\frac{dr_2^{j*}}{dr_1} = 0$. Differentiating both sides with respect to r_1,

$$W_{11}^j(r_1,I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_2^{j*}, I) \end{array}\right\} = W_1^j(r_1,I)\left\{C_1(r_2^{j*})\right\} + \frac{d\mu_2^*}{dr_1}. \tag{6}$$

From the assumptions on $W^j(\cdot,\cdot)$ and $C(\cdot)$, I find that $\frac{d\mu_2^*}{dr_1} < 0$. This means that if there is no war chest, if one decreases r_1, there will still be no war chest. Conversely, if one increases r_1, there may be a war chest (as one hits the corner where $\mu_2^* = 0$, one moves into Case 4).

[2] Thus, $r_L < \bar{r} < r_H$.

A.2.4 Case 4:

Here, $\mu_1 = 0$ and $\mu_2 = 0$. This implies that $r_1 > s_1^j > 0$. This yields the first-order condition:

$$W_1^j(s_1^j, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^j + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^j + r_2^{j*}, I) \end{array}\right\} = W^j(s_1^j, I)\left\{C_1(r_2^{j*})\right\}. \quad (7)$$

Since the term in braces on the LHS is greater than 1, and $W^j(s_1^j, I) < 1$, $C_1(r_2^{j*}) > W_1^j(s_1^j, I)$. Substituting in the solution value of s_1^j,

$$W_1^j(s_1^{j*}, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^{j*} + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^{j*} + r_2^{j*}, I) \end{array}\right\} \equiv W^j(s_1^{j*}, I)\left\{C_1(r_2^{j*})\right\}. \quad (8)$$

Differentiating both sides with respect to r_1, and substituting in Equation (3) to simplify,

$$\begin{array}{l} W_{11}^j(s_1^{j*}, I)\left\{\begin{array}{c} 1 - C(r_2^{j*}) + \eta_2 W^H(r_1 - s_1^{j*} + r_2^{j*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^{j*} + r_2^{j*}, I) \end{array}\right\}\dfrac{ds_1^{j*}}{dr_1} \\ = W_1^j(s_1^{j*}, I)C_1(r_2^{j*})\left[2\dfrac{ds_1^{j*}}{dr_1} - 1\right] + W^j(s_1^{j*}, I)C_{11}(r_2^{j*})\dfrac{dr_2^{j*}}{dr_1}. \end{array} \quad (9)$$

From Equation (4), if $\frac{dr_2^{j*}}{dr_1} > 0$, then $\frac{ds_1^{j*}}{dr_1} > 1$. This yields a contradiction. Thus, $\frac{dr_2^{j*}}{dr_1} < 0$, which means that $\frac{ds_1^{j*}}{dr_1} < 1$. If $\frac{ds_1^{j*}}{dr_1} < 0$, this yields another contradiction. Thus, $0 < \frac{ds_1^{j*}}{dr_1} < 1$.

A.3 Solving for money raised in the first election

Now solve the grand maximization problem:

$$\begin{array}{lll} \max\limits_{r_1 \geq 0} & -C(r_1) & +\eta_1 \left\{W^H(s_1^{H*}, I)\left[\begin{array}{c} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array}\right]\right\} \\ & & +(1-\eta_1)\left\{W^L(s_1^{L*}, I)\left[\begin{array}{c} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array}\right]\right\} \end{array}$$

where $0 < s_1^{L*} \leq r_1$ and $0 < s_1^{H*} \leq r_1$. Using the method of Kuhn-Tucker, set up the Lagrangian:

$$\begin{array}{ll} L(r_1, \nu) = & -C(r_1) \\ & +\eta_1 \left\{W^H(s_1^{H*}, I)\left[\begin{array}{c} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array}\right]\right\} \\ & +(1-\eta_1)\left\{W^L(s_1^{L*}, I)\left[\begin{array}{c} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2)W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array}\right]\right\} \\ & +\nu r_1. \end{array}$$

The first-order conditions are (again, substituting with Equation (3))

$$\begin{aligned}
\frac{dL}{dr_1} = & -C_1(r_1) \\
& +\eta_1 \left\{ \begin{array}{l} W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \frac{ds_1^{H*}}{dr_1} \\ +W^H(s_1^{H*}, I) C_1(r_2^{H*}) \left[1 - \frac{ds_1^{H*}}{dr_1} \right] \end{array} \right\} \\
& +(1-\eta_1) \left\{ \begin{array}{l} W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \frac{ds_1^{L*}}{dr_1} \\ +W^L(s_1^{H*}, I) C_1(r_2^{L*}) \left[1 - \frac{ds_1^{L*}}{dr_1} \right] \end{array} \right\} \\
& +\nu \\
= & \ 0
\end{aligned}$$

$$\nu r_1 = 0, \quad \nu \geq 0, \quad r_1 \geq 0.$$

There are two cases: 1-$\{\nu > 0\}$and 2-$\{\nu = 0\}$.

A.3.1 Case 1:

Since $\nu > 0$, then $r_1 = 0$, and thus $s_1^{H*} = s_1^{L*} = 0$ (and $r_2^{H*} = r_2^{L*}$). For this to be a maximum

$$\begin{aligned}
& -C(0) + \left\{ \eta_1 W^H(0, I) + (1-\eta_1) W^L(0, I) \right\} \left[\begin{array}{r} 1 - C(r_2^*) + \eta_2 W^H(r_2^*, I) \\ +(1-\eta_2) W^L(r_2^*, I) \end{array} \right] \geq \\
& -C(r_1) + \left\{ \eta_1 W^H(r_1, I) + (1-\eta_1) W^L(r_1, I) \right\} \left[\begin{array}{r} 1 - C(r_2^*) + \eta_2 W^H(r_2^*, I) \\ +(1-\eta_2) W^L(r_2^*, I) \end{array} \right]
\end{aligned}$$

for all $r_1 = s_1^{H*} = s_1^{L*}$.[3] Since the term in brackets is greater than 1, if $r_1 = \varepsilon$, the inequality will not hold as a result of the Inada-type conditions on $W^j(\cdot, \cdot)$ and $C(\cdot)$. This also rules out the possibility of Case 1 in the second maximization problem (in Section A.2).

A.3.2 Case 2:

This is the only possibility that allows $r_1 > 0$. Thus, $r_1^* > 0$.

A.4 Solution Maximum

I show that this problem satisfies the conditions of the Theorem of Kuhn and Tucker under Convexity (Sundaram 1996, Theorem 7.16), and thus, that r_1^* is the maximum. Let

$$\begin{aligned}
g(r_1) = \ & -C(r_1) \quad +\eta_1 \left\{ W^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \right\} \\
& +(1-\eta_1) \left\{ W^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \right\}
\end{aligned}$$

[3] It must also be greater for all $r_1 > s_1^{H*}$, etc., but the inequality stated will be sufficient to demonstrate the contradiction.

so that the incumbent maximizes $g(r_1)$. If $[g'(\hat{r}_1) - g'(r_1)](\hat{r}_1 - r_1) \leq 0$ for all $\hat{r}_1, r_1 \geq 0$, then g is concave. Using the first derivative from the Lagrangian on r_1,

$$
\begin{aligned}
g'(r_1) = \quad & -C_1(r_1) \\
& +\eta_1 \left\{ \begin{array}{l} W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \dfrac{ds_1^{H*}}{dr_1} \\ +W^H(s_1^{H*}, I) C_1(r_2^{H*}) \left[1 - \dfrac{ds_1^{H*}}{dr_1} \right] \end{array} \right\} \\
& +(1-\eta_1) \left\{ \begin{array}{l} W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \dfrac{ds_1^{L*}}{dr_1} \\ +W^L(s_1^{H*}, I) C_1(r_2^{L*}) \left[1 - \dfrac{ds_1^{L*}}{dr_1} \right] \end{array} \right\}
\end{aligned}
$$

If the first challenger is high quality, $r_1 = s_1^{H*}$, and thus $\dfrac{ds_1^{H*}}{dr_1} = 1$. If the first challenger is low quality, $r_1 \geq s_1^{L*}$. If $r_1 > s_1^{L*}$,Equation (8) holds. If $r_1 = s_1^{L*}$, then the equation is simplified. Substituting these in, we obtain

$$
\begin{aligned}
g'(r_1) = \quad & -C_1(r_1) \\
& +\eta_1 W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \\
& +(1-\eta_1) \left\{ \begin{array}{l} W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \dfrac{ds_1^{L*}}{dr_1} \\ +W^L(s_1^{H*}, I) C_1(r_2^{L*}) \left[1 - \dfrac{ds_1^{L*}}{dr_1} \right] \end{array} \right\}.
\end{aligned}
$$

We finally obtain

$$
\begin{aligned}
g'(r_1) = \quad -C_1(r_1) \quad & +\eta_1 W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \\
& +(1-\eta_1) W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right]
\end{aligned} \tag{10}
$$

Let

$$
\begin{aligned}
g'(\hat{r}_1) = \quad -C_1(\hat{r}_1) \quad & +\eta_1 W_1^H(\hat{s}_1^{H*}, I) \left[\begin{array}{r} 1 - C(\hat{r}_2^{H*}) + \eta_2 W^H(\hat{r}_1 - \hat{s}_1^{H*} + \hat{r}_2^{H*}, I) \\ +(1-\eta_2) W^L(\hat{r}_1 - \hat{s}_1^{H*} + \hat{r}_2^{H*}, I) \end{array} \right] \\
& +(1-\eta_1) W_1^L(\hat{s}_1^{L*}, I) \left[\begin{array}{r} 1 - C(\hat{r}_2^{L*}) + \eta_2 W^H(\hat{r}_1 - \hat{s}_1^{L*} + \hat{r}_2^{L*}, I) \\ +(1-\eta_2) W^L(\hat{r}_1 - \hat{s}_1^{L*} + \hat{r}_2^{L*}, I) \end{array} \right]
\end{aligned}
$$

Without loss of generality, let $\hat{r}_1 > r_1$. (A) Since $C_{11}(r) > 0$, $-C_1(\hat{r}_1) < -C_1(r_1)$. (B) Since $W_{11}(s, I) < 0$ and $\dfrac{ds_1^{j*}}{dr_1} > 0$, $W_1(\hat{s}_1^{j*}, I) < W_1(s_1^{j*}, I)$. (C) Since $C_1(r) > 0$ and $\dfrac{dr_2^{j*}}{dr_1} \leq 0$, $C(\hat{r}_2^{j*}) \leq C(r_2^{j*})$. (D) Since $W_1(s, I) > 0$ and $\dfrac{d}{dr_1}(r_1 - s_1^{q_1*} + r_2^{q_1*}) \geq 0$ (see Equation 4), $W(\hat{r}_1 - \hat{s}_1^{j*} + \hat{r}_2^{j*}, I) \geq W(r_1 - s_1^{j*} + r_2^{j*}, I)$.

(E) From (B), (C), and (D),

$$\left\{ \begin{array}{l} \eta_1 W_1^H(\hat{s}_1^{H*}, I) \left[\begin{array}{r} 1 - C(\hat{r}_2^{H*}) + \eta_2 W^H(\hat{r}_1 - \hat{s}_1^{H*} + \hat{r}_2^{H*}, I) \\ +(1-\eta_2) W^L(\hat{r}_1 - \hat{s}_1^{H*} + \hat{r}_2^{H*}, I) \end{array} \right] \\ +(1-\eta_1) W_1^L(\hat{s}_1^{L*}, I) \left[\begin{array}{r} 1 - C(\hat{r}_2^{L*}) + \eta_2 W^H(\hat{r}_1 - \hat{s}_1^{L*} + \hat{r}_2^{L*}, I) \\ +(1-\eta_2) W^L(\hat{r}_1 - \hat{s}_1^{L*} + \hat{r}_2^{L*}, I) \end{array} \right] \end{array} \right\}$$
$$\leq \left\{ \begin{array}{l} \eta_1 W_1^H(s_1^{H*}, I) \left[\begin{array}{r} 1 - C(r_2^{H*}) + \eta_2 W^H(r_1 - s_1^{H*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{H*} + r_2^{H*}, I) \end{array} \right] \\ +(1-\eta_1) W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1 - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1 - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \end{array} \right\}.$$

From (A) and (E), we obtain $g'(\hat{r}_1) - g'(r_1) < 0$. Since $\hat{r}_1 > r_1$, we have $[g'(\hat{r}_1) - g'(r_1)](\hat{r}_1 - r_1) \leq 0$ for all $\hat{r}_1, r_1 \geq 0$. The constraint $h(r_1) = r_1 \geq 0$ is concave, and has a point $h(\overline{r}_1) > 0$ (specifically, any $r_1 > 0$), thus fulfilling Slater's condition. Since the other Kuhn-Tucker first-order conditions hold, r_1^* is the maximum.□

A.5 The possibility of a war chest

Now I am left to show when a war chest may exist, *i.e.*, when $r_1^* > s_1^{L*}$ and/or $r_1^* > s_1^{H*}$. To show this, first solve for r_1 if the incumbent knew for certain that he would face a low quality challenger in the first election. Call this r_1^L. I now have the maximization problem:

$$\max_{r_1^L \geq 0} -C(r_1^L) + W^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1^L - s_1^{L*} + r_2^{H*}, I) \\ +(1-\eta_2) W^L(r_1^L - s_1^{L*} + r_2^{L*}, I) \end{array} \right].$$

Through the Kuhn-Tucker method, obtain the Lagrangian:

$$L(r_1^L, \nu^L) = -C(r_1^L) + W^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1^L - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1^L - s_1^{L*} + r_2^{L*}, I) \end{array} \right] + \nu^L r_1^L.$$

The first-order conditions are

$$\begin{aligned} \frac{dL}{dr_1^L} = & -C_1(r_1^L) \\ & + W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1^L - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1^L - s_1^{L*} + r_2^{L*}, I) \end{array} \right] \frac{ds_1^{L*}}{dr_1^L} \\ & + W^L(s_1^{L*}, I) C_1(r_2^{L*}) \left[1 - \frac{ds_1^{L*}}{dr_1^L} \right] \\ & + \nu^L \\ = & \; 0 \end{aligned} \tag{11}$$

$$\nu^L r_1^L = 0, \quad \nu^L \geq 0, \quad r_1^L \geq 0.$$

As before, I can rule out $r_1^L = 0$, which implies $\nu^L = 0$. If (as in Case 3 of Section A.2 above) $r_1^L = s_1^{L*}$, then $\dfrac{ds_1^{L*}}{dr_1^L} = 1$, and Equation (11) reduces to

$$C_1(r_1^L) = W_1^L(s_1^{L*}, I) \left[\begin{array}{r} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1^L - s_1^{L*} + r_2^{L*}, I) \\ +(1-\eta_2) W^L(r_1^L - s_1^{L*} + r_2^{L*}, I) \end{array} \right]. \tag{12}$$

This makes sense in that the marginal cost in this election must be equal to the marginal benefit of winning this election *plus* the expected utility of the next election. And if (as in Case 4 of Section A.2 above) $r_1^L > s_1^{L*}$, then (from Equation (7)):

$$W_1^L(s_1^{L*}, I)\left[\begin{array}{c} 1 - C(r_2^{L*}) + \eta_2 W^H(r_1^L - s_1^{L*} + r_2^{H*}, I) \\ +(1-\eta_2)W^L(r_1^L - s_1^{L*} + r_2^{L*}, I) \end{array}\right] = W^L(s_1^{L*}, I)C_1(r_2^{L*}). \tag{13}$$

Substituting Equation (13) into Equation (11), I obtain Equation (12) again. If $r_1^L > s_1^{L*}$, substitute Equation (12) into Equation (13) to get

$$C_1(r_1^L) = W^L(s_1^{L*}, I)C_1(r_2^{L*}). \tag{14}$$

Since $W^L(s_1^{L*}, I) < 1$, this requires that $C_1(r_2^{L*}) > C_1(r_1^L)$, and thus, $r_2^{L*} > r_1^L$ (if $r_1^L > s_1^{L*}$).

Going through the same process to find r_1^H and its accompanying conditions, if $r_1^H > s_1^{H*}$, then

$$C_1(r_1^H) = W^H(s_1^{H*}, I)C_1(r_2^{H*})$$

and $r_2^{H*} > r_1^H$. Since by assumption, $r_L > r_H$ (the monies raised and spent if there were only one election and the incumbent knew the quality of his challenger was low or high, respectively), then the result (from Equation (7)) that $C_1(r_2^{L*}) > W_1^L(s_1^{L*}, I)$ yields $s_1^{L*} > r_2^{L*}$. But this means that $s_1^{L*} > r_2^{L*} > r_1^L$, and since $s_1^{L*} \leq r_1^L$, this yields a contradiction. Therefore, $r_1^L = s_1^{L*}$: an incumbent would never have a war chest if he knew that he was facing a high quality challenger in this election.

Now I show that $r_1^H < r_1^L$. There are two cases: 1-$\{r_1^H = s_1^{H*}\}$and 2-$\{r_1^H > s_1^{H*}\}$ (knowing that $r_1^L = s_1^{L*}$).

A.5.1 Case 1:

I wish to show that $r_1^H < r_1^L$ when $r_1^H = s_1^{H*}$. Suppose not. Then $r_1^H \geq r_1^L$, which implies that $C_1(r_1^H) \geq C_1(r_1^L)$. Substituting in the reduced first-order conditions from Equation (12),

$$\begin{gathered} W_1^H(r_1^H, I)\left[1 - C(r_2^*) + \eta_2 W^H(r_2^*, I) + (1-\eta_2)W^L(r_2^*, I)\right] \geq \\ W_1^L(r_1^L, I)\left[1 - C(r_2^*) + \eta_2 W^H(r_2^*, I) + (1-\eta_2)W^L(r_2^*, I)\right] \end{gathered}$$

where $r_2^* = \bar{r}$ of Section A.2.2. Canceling out:

$$W_1^H(r_1^H, I) \geq W_1^L(r_1^L, I).$$

This yields a contradiction to the assumption that $r_L > r_H$.[4] Thus, $r_1^H < r_1^L$ (if there is no war chest).

A.5.2 Case 2:

I wish to show that $r_1^H < r_1^L$ when $r_1^H > s_1^{H*}$. Since $r_1^L = s_1^{L*}$, the incumbent brings no war chest with him into the second election cycle (if he draws a low quality challenger in the first election), and $r_2^{L*} = \bar{r}$. From Equation (12), $C_1(r_1^L) > W_1^L(r_1^L, I)$. This implies that $r_1^L > r_L > \bar{r}$. Since $r_1^H > s_1^{H*}$, $r_2^{H*} < \bar{r}$. From Equation (14), $r_2^{H*} > r_1^H$, and thus $r_1^H < \bar{r}$. Therefore $r_1^H < r_1^L$.

Since $\eta \in (0,1)$, by the concavity of the overall utility function, one obtains $r_1^H < r_1^* < r_1^L$. Now I examine when to expect a war chest. If the incumbent knew he was running against a low quality challenger, he would raise r_1^L (and spend $s_1^{L*} = r_1^L$). From the point of view of the model, he is in Case 3, and thus, $\mu_2 > 0$. Since $\dfrac{d\mu_2^*}{dr_1} < 0$ (from Equation (6)), as one decreases r_1 from r_1^L to r_1^*, μ_2^* increases, and thus, there

[4] Or, more specifically, this yields a contradition to the shapes that $W^L(\cdot)$ and $W^H(\cdot)$ must take to maintain the assumption that $r_L > r_H$.

is no war chest if an incumbent found out he was running against a low quality challenger *after* he raised the money. If the incumbent knew he was running against a high quality challenger, he would raise r_1^H. He may save some or none of it in this case. If he had no war chest in this case, since $\frac{d\mu_2^*}{dr_1} < 0$, as one increases r_1 from r_1^H to r_1^*, μ_2^* decreases—perhaps reaching 0, in which case there would be a war chest. If the incumbent had a war chest at r_1^H, then since $\frac{ds_1^{H*}}{dr_1^H} < 1$ (from Equation (9)), as one increased r_1 from r_1^H to r_1^*, the incumbent would keep a larger and larger war chest.□

A.6 Relationship between incumbent strength and money

I utilize an application of the implicit function theorem for simultaneous equations (Chiang 1984, 210-212). The equations used are the first order conditions derived above: Equations 2 (twice: once for each challenger quality), 8 (twice: once for each challenger quality), and 10 (substituting in $r_1 = r_1^*$ to form the identity and setting it equal to 0). Dropping the * for simplicity:

$$\begin{array}{lll}
F^1: & -C_1(r_2^H) + \eta_2 W_1^H(r_1 - s_1^H + r_2^H, I) + (1-\eta_2) W_1^L(r_1 - s_1^H + r_2^H, I) & = 0 \\
F^2: & -C_1(r_2^L) + \eta_2 W_1^H(r_1 - s_1^L + r_2^L, I) + (1-\eta_2) W_1^L(r_1 - s_1^L + r_2^L, I) & = 0 \\
F^3: & W_1^H(s_1^H, I)\left\{\begin{array}{c} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array}\right\} - W^H(s_1^H, I)\left\{C_1(r_2^H)\right\} & = 0 \\
F^4: & W_1^L(s_1^L, I)\left\{\begin{array}{c} 1 - C(r_2^L) + \eta_2 W^H(r_1 - s_1^L + r_2^L, I) \\ +(1-\eta_2) W^L(r_1 - s_1^L + r_2^L, I) \end{array}\right\} - W^L(s_1^L, I)\left\{C_1(r_2^L)\right\} & = 0 \\
F^5: & \left\{\begin{array}{ll} -C_1(r_1) & +\eta_1 W_1^H(s_1^H, I)\left[\begin{array}{c} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array}\right] \\ & +(1-\eta_1) W_1^L(s_1^L, I)\left[\begin{array}{c} 1 - C(r_2^L) + \eta_2 W^H(r_1 - s_1^L + r_2^L, I) \\ +(1-\eta_2) W^L(r_1 - s_1^L + r_2^L, I) \end{array}\right] \end{array}\right\} & = 0
\end{array}$$

Assuming continuous partial derivatives and a nonzero Jacobian determinant, through the implict function theorem, we derive the following system:

$$\begin{bmatrix}
\frac{\partial F^1}{\partial r_1} & \frac{\partial F^1}{\partial s_1^H} & \frac{\partial F^1}{\partial s_1^L} & \frac{\partial F^1}{\partial r_2^H} & \frac{\partial F^1}{\partial r_2^L} \\
\frac{\partial F^2}{\partial r_1} & \frac{\partial F^2}{\partial s_1^H} & \frac{\partial F^2}{\partial s_1^L} & \frac{\partial F^2}{\partial r_2^H} & \frac{\partial F^2}{\partial r_2^L} \\
\frac{\partial F^3}{\partial r_1} & \frac{\partial F^3}{\partial s_1^H} & \frac{\partial F^3}{\partial s_1^L} & \frac{\partial F^3}{\partial r_2^H} & \frac{\partial F^3}{\partial r_2^L} \\
\frac{\partial F^4}{\partial r_1} & \frac{\partial F^4}{\partial s_1^H} & \frac{\partial F^4}{\partial s_1^L} & \frac{\partial F^4}{\partial r_2^H} & \frac{\partial F^4}{\partial r_2^L} \\
\frac{\partial F^5}{\partial r_1} & \frac{\partial F^5}{\partial s_1^H} & \frac{\partial F^5}{\partial s_1^L} & \frac{\partial F^5}{\partial r_2^H} & \frac{\partial F^5}{\partial r_2^L}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial r_1}{\partial I} \\ \frac{\partial s_1^H}{\partial I} \\ \frac{\partial s_1^L}{\partial I} \\ \frac{\partial r_2^H}{\partial I} \\ \frac{\partial r_2^L}{\partial I}
\end{bmatrix}
=
\begin{bmatrix}
-\frac{\partial F^1}{\partial I} \\ -\frac{\partial F^2}{\partial I} \\ -\frac{\partial F^3}{\partial I} \\ -\frac{\partial F^4}{\partial I} \\ -\frac{\partial F^5}{\partial I}
\end{bmatrix}.$$

The signs of the elements of the Jacobian are generally easily found:

$$\begin{bmatrix}
- & + & 0 & - & 0 \\
- & 0 & + & 0 & - \\
+ & - & 0 & - & 0 \\
+ & 0 & - & 0 & - \\
? & - & - & 0 & 0
\end{bmatrix}.$$

The one unknown, $\frac{\partial F^5}{\partial r_1}$, can be signed as negative given the assumption of $C_{11}(r) > [C_1(r)]^2$ (this is a sufficient, not necessary, condition). From this, the Jacobian determinant is negative. We can now find the sign of $\frac{\partial r_1}{\partial I}$ through Cramer's rule:

$$\frac{\partial r_1}{\partial I} = \frac{\begin{vmatrix} -\frac{\partial F^1}{\partial I} & \frac{\partial F^1}{\partial s_1^H} & \frac{\partial F^1}{\partial s_1^L} & \frac{\partial F^1}{\partial r_2^H} & \frac{\partial F^1}{\partial r_2^L} \\ -\frac{\partial F^2}{\partial I} & \frac{\partial F^2}{\partial s_1^H} & \frac{\partial F^2}{\partial s_1^L} & \frac{\partial F^2}{\partial r_2^H} & \frac{\partial F^2}{\partial r_2^L} \\ -\frac{\partial F^3}{\partial I} & \frac{\partial F^3}{\partial s_1^H} & \frac{\partial F^3}{\partial s_1^L} & \frac{\partial F^3}{\partial r_2^H} & \frac{\partial F^3}{\partial r_2^L} \\ -\frac{\partial F^4}{\partial I} & \frac{\partial F^4}{\partial s_1^H} & \frac{\partial F^4}{\partial s_1^L} & \frac{\partial F^4}{\partial r_2^H} & \frac{\partial F^4}{\partial r_2^L} \\ -\frac{\partial F^5}{\partial I} & \frac{\partial F^5}{\partial s_1^H} & \frac{\partial F^5}{\partial s_1^L} & \frac{\partial F^5}{\partial r_2^H} & \frac{\partial F^5}{\partial r_2^L} \end{vmatrix}}{\begin{vmatrix} \frac{\partial F^1}{\partial r_1} & \frac{\partial F^1}{\partial s_1^H} & \frac{\partial F^1}{\partial s_1^L} & \frac{\partial F^1}{\partial r_2^H} & \frac{\partial F^1}{\partial r_2^L} \\ \frac{\partial F^2}{\partial r_1} & \frac{\partial F^2}{\partial s_1^H} & \frac{\partial F^2}{\partial s_1^L} & \frac{\partial F^2}{\partial r_2^H} & \frac{\partial F^2}{\partial r_2^L} \\ \frac{\partial F^3}{\partial r_1} & \frac{\partial F^3}{\partial s_1^H} & \frac{\partial F^3}{\partial s_1^L} & \frac{\partial F^3}{\partial r_2^H} & \frac{\partial F^3}{\partial r_2^L} \\ \frac{\partial F^4}{\partial r_1} & \frac{\partial F^4}{\partial s_1^H} & \frac{\partial F^4}{\partial s_1^L} & \frac{\partial F^4}{\partial r_2^H} & \frac{\partial F^4}{\partial r_2^L} \\ \frac{\partial F^5}{\partial r_1} & \frac{\partial F^5}{\partial s_1^H} & \frac{\partial F^5}{\partial s_1^L} & \frac{\partial F^5}{\partial r_2^H} & \frac{\partial F^5}{\partial r_2^L} \end{vmatrix}}.$$

The determinant in the numerator has the following signs:

$$\begin{vmatrix} - & + & 0 & - & 0 \\ - & 0 & + & 0 & - \\ ? & - & 0 & - & 0 \\ ? & 0 & - & 0 & - \\ - & - & - & 0 & 0 \end{vmatrix}.$$

Through manipulation and substitution of the actual partial derivatives of the equations, it can be shown that the determinant is negative irrespective of the unknown signs. Thus, we find that $\frac{\partial r_1}{\partial I} > 0$. Through similar applications of Cramer's rule, it can be shown that all of the comparative statics in this system ($\frac{\partial s_1^H}{\partial I}, \frac{\partial r_2^L}{\partial I}$, etc.) are positive. A similar exercise can be conducted for when there are no war chests (in which case the number of equations reduces to three).□

A.7 Relationship between incumbent strength and war chests

In this model, the war chest is $r_1 - s_1^j$. Thus, we wish to find the sign of $\frac{\partial (r_1 - s_1^j)}{\partial I}$ or $\frac{\partial r_1}{\partial I} - \frac{\partial s_1^j}{\partial I}$. A sufficient, but not necessary, condition for this is that $\frac{\partial F^3}{\partial I}$ and $\frac{\partial F^4}{\partial I}$ both be negative. We find that

$$\begin{aligned} \frac{\partial F^3}{\partial I} = \; & W_{12}^H(s_1^H, I) \left\{ \begin{array}{c} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \\ & + W_1^H(s_1^H, I) \left\{ \begin{array}{c} \eta_2 W_2^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W_2^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \\ & - W_2^H(s_1^H, I) \left\{ C_1(r_2^H) \right\}. \end{aligned}$$

Since the first two two terms are positive, and the third negative, we need to show that

$$\begin{aligned} W_2^H(s_1^H, I) \left\{ C_1(r_2^H) \right\} > \; & W_{12}^H(s_1^H, I) \left\{ \begin{array}{c} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \\ & + W_1^H(s_1^H, I) \left\{ \begin{array}{c} \eta_2 W_2^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W_2^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} \end{aligned}$$

or

$$C_1(r_2^H) > \frac{W_{12}^H(s_1^H, I)}{W_2^H(s_1^H, I)} \left\{ \begin{array}{c} 1 - C(r_2^H) + \eta_2 W^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\} + W_1^H(s_1^H, I) \frac{\left\{ \begin{array}{c} \eta_2 W_2^H(r_1 - s_1^H + r_2^H, I) \\ +(1-\eta_2) W_2^L(r_1 - s_1^H + r_2^H, I) \end{array} \right\}}{W_2^H(s_1^H, I)}$$

From the assumption $W_2(s_1, I) >> W_{12}(s_1, I)$, the first term on the right-hand side is close to zero. From the arguments above, $s_1^H > r_1 - s_1^H + r_2^H$, which implies that the fraction of the second line is less than one. Since $C_1(r_2^H) > W_1^H(s_1^H, I)$, the inequality holds. A similar argument can be made for $\frac{\partial F^4}{\partial I}$.□

B Proof of Proposition 2

In this section, I give a more detailed proof of Proposition 2: the two-election equilibrium with no uncertainty. As a reminder, the incumbent spends all he has in the last election.

Step 1: β^***.** The challenger runs if $c^H < 1 - W^H(r_1 - s_1^j + r_2^j, I)$ and does not run if $c^H \geq 1 - W^H(r_1 - s_1^j + r_2^j, I)$.[5]

Step 2: σ^***.** Having raised r_1 and knowing the first election challenger j ($= L$ or H), the incumbent must choose s_1^j and r_2^j. The incumbent can either save and raise enough to deter the challenger in the second election or not. If the incumbent does not try to deter, he maximizes the following:

$$\max_{s_1^j, r_2^j} W^j(s_1^j, I) \left[1 - C(r_2^j) + W^H(r_1 - s_1^j + r_2^j, I) \right].$$

If he tries to deter, he maximizes the following:

$$\max_{s_1^j, r_2^j} W^j(s_1^j, I) \left[1 - C(r_2^j) + W^L(r_1 - s_1^j + r_2^j, I) \right]$$

$$\text{such that } 1 - W^H(r_1 - s_1^j + r_2^j, I) \leq c^H.$$

The incumbent chooses the strategy which yields the greater utility. Because of the assumption on $W^j(s, I)$ and the comparative statics above, weaker incumbents must save and raise more to meet the deterrence constraint. Depending on r_1, there will be up to three regions. In the lower region, incumbents find it too costly to deter, and maximize the first argument. In the middle region, incumbents deter the challenger, and the constraint is binding (so that $1 - W^H(r_1 - s_1^j + r_2^j, I) = c^H$), which means that the incumbent saves and raises extra money to deter the challenger. In the upper region, incumbents deter the challenger, but the constraint is not binding, so no extra money is raised.

Step 3: α^***.** The challenger runs if $c^H < 1 - W^H(s_1^H, I)$ and does not run if $c^H \geq 1 - W^H(s_1^H, I)$.

Step 4: ρ^***.** The incumbent must choose r_1. The incumbent can either raise enough to deter the challenger in the first election or not. If the incumbent does not try to deter in either election, he maximizes the following:

$$\max_{r_1} -C(r_1) + W^H(s_1^H(r_1), I) \left\{ 1 - C(r_2^H(r_1)) + W^H(r_1 - s_1^H(r_1) + r_2^H(r_1), I) \right\}$$

where $s_1^H(r_1)$ denotes that the s_1^H chosen by the incumbent depends on r_1 (and similarly for r_2^H). If the incumbent does not try to deter in the first election, but will try to deter in the second eleciton, he maximizes the following:

$$\max_{r_1} -C(r_1) + W^H(s_1^H(r_1), I) \left\{ 1 - C(r_2^H(r_1)) + W^L(r_1 - s_1^H(r_1) + r_2^H(r_1), I) \right\}$$

[5] If the challenger entered when indifferent, then there would be an open-set problem for the incumbent.

$$\text{such that } 1 - W^H(r_1 - s_1^L + r_2^L, I) \leq c^H \ .$$

If the incumbent does try to deter, he maximizes the following:[6]

$$\max_{r_1} -C(r_1) + W^L(s_1^L(r_1), I)\left\{1 - C(r_2^L(r_1)) + W^L(r_1 - s_1^L(r_1) + r_2^L(r_1), I)\right\}$$

$$\begin{aligned} \text{such that} \quad & 1 - W^H(s_1^H, I) \leq c^H \\ \text{and} \quad & 1 - W^H(r_1 - s_1^L + r_2^L, I) \leq c^H \ . \end{aligned}$$

The incumbent chooses whether to deter by which approach gives him the most utility. There are up to four regions. In the lowest region, incumbents find it too costly to deter, and maximize the first argument. In the lower-middle region, incumbents do not deter in the first election, but do deter in the second election, and that constraint is binding ($1 - W^H(r_1 - s_1^L + r_2^L, I) = c^H$). In the upper-middle region, incumbents deter the challenger, and the constraint is binding (so that $1 - W^H(s_1^H, I) = c^H$ and $1 - W^H(r_1 - s_1^L + r_2^L, I) = c^H$).[7] In the highest region, incumbents deter the challenger, but the constraint is not binding, so no extra money is raised.□

References

[1] Chiang, Alpha C. 1984. *Fundamental Methods of Mathematical Economics*, 3rd edition. New York: McGraw-Hill/Irwin.

[2] Sundaram, Rangarajan K. 1996. *A First Course in Optimization Theory.* New York: Cambridge University Press.

[6] It is straightforward, but tedious, to show that if the incumbent chooses to deter the incumbent in the first election, he will also try to deter in the second election.

[7] Actually, for the strongest incumbents in this middle region, the first constraint is not binding, but the second constraint is.